Course Outcome 2 (CO2)

Students should be able to understand and evaluate one-dimensional heat flow and in different geometries

Lesson Outcomes from CO2 (Part 1)

- To **derive** the equation for temperature distribution in various geometries
- Thermal Resistance concept – to **derive** expression for various geometries
- To **evaluate** the heat transfer using thermal resistance in various geometries
- To **evaluate** the critical radius of insulation
- To **evaluate** heat transfer from the rectangular fins

BOUNDARY AND INITIAL CONDITIONS

A heat transfer problem must equipped with a description of the thermal conditions at the bounding surfaces of the medium.

Boundary conditions: The *mathematical expressions* of the thermal conditions at the boundaries.

The **initial condition** specified at time ***t* = 0**, , which is a mathematical expression for the temperature distribution of the medium initially.

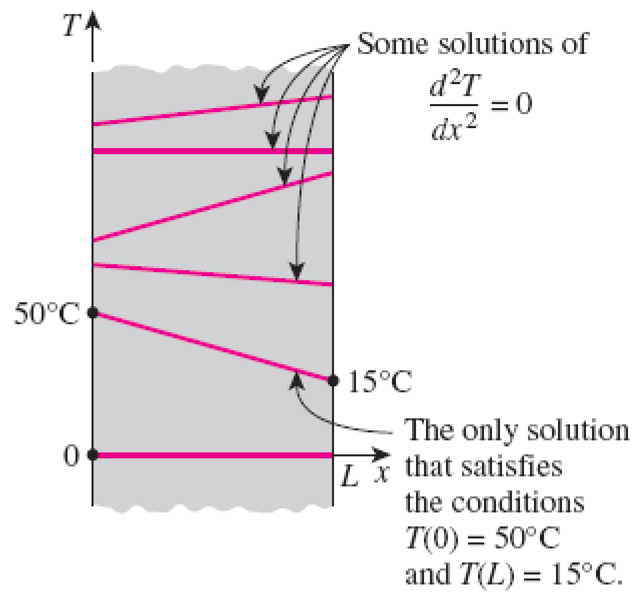


FIGURE 2–26

To describe a heat transfer problem completely, two boundary conditions must be given for each direction along which heat transfer is significant.

Boundary Conditions

- Specified Temperature Boundary Condition
- Specified Heat Flux Boundary Condition
- Convection Boundary Condition
- Radiation Boundary Condition
- Interface Boundary Conditions
- Generalized Boundary Conditions

Boundary Conditions

- Specified Temperature Boundary Condition
- Specified Heat Flux Boundary Condition
- Convection Boundary Condition
- Radiation Boundary Condition
- Interface Boundary Conditions
- Generalized Boundary Conditions

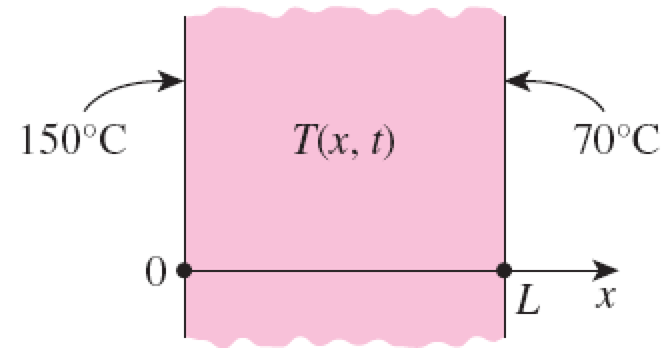
1 Specified Temperature Boundary Condition

For one-dimensional heat transfer through a plane wall of thickness L, the specified temperature boundary conditions can be expressed as

$$T(0, t) = T_1$$
$$T(L, t) = T_2$$

where T_1 and T_2 are the specified temperatures at surfaces at $x = 0$ and $x = L$, respectively.

The specified temperatures can be constant, which is the case for steady heat conduction, or may vary with time.



$$T(0, t) = 150°\text{C}$$
$$T(L, t) = 70°\text{C}$$

FIGURE 2–27

Specified temperature boundary conditions on both surfaces of a plane wall.

Boundary Conditions

- Specified Temperature Boundary Condition
- Specified Heat Flux Boundary Condition
- Convection Boundary Condition
- Radiation Boundary Condition
- Interface Boundary Conditions
- Generalized Boundary Conditions

2 Specified Heat Flux Boundary Condition

The heat flux in the positive *x*-direction anywhere in the medium, including the boundaries

$$\dot{q} = -k\frac{\partial T}{\partial x} = \left(\begin{array}{c}\text{Heat flux in the} \\ \text{positive } x - \text{direction}\end{array}\right) \qquad (\text{W/m}^2)$$

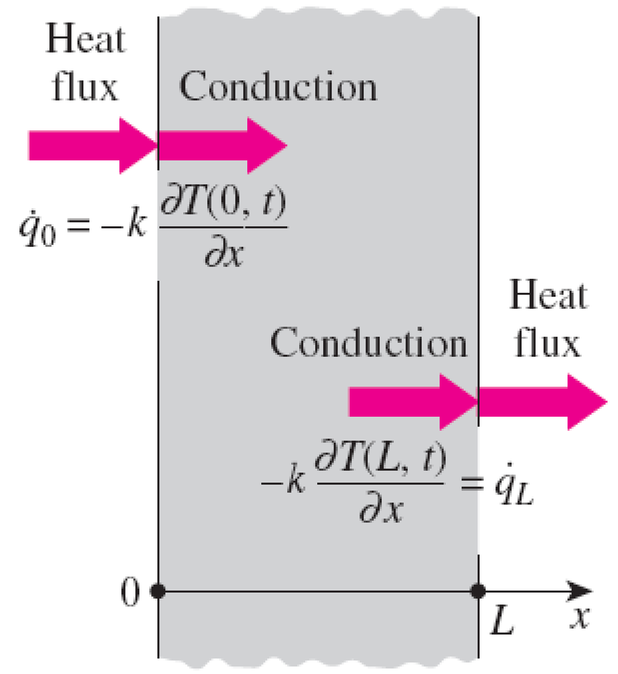


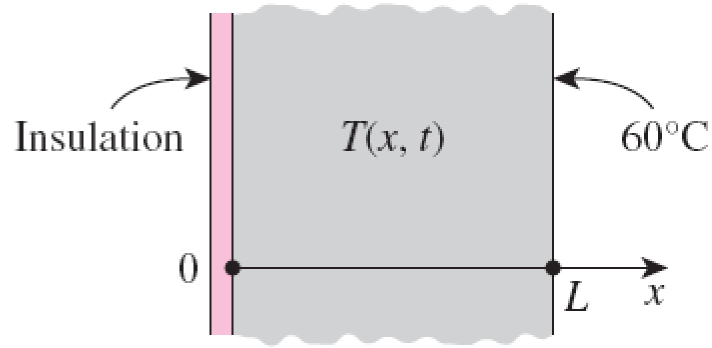
FIGURE 2–28

Specified heat flux boundary conditions on both surfaces of a plane wall.

Special Case 1: Insulated Boundary

A well-insulated surface is a surface with heat flux is zero. Then the boundary condition on a perfectly insulated surface (at x = 0,) can be expressed as

$$k\frac{\partial T(0,t)}{\partial x}=0 \qquad \text{or} \qquad \frac{\partial T(0,t)}{\partial x}=0$$



$$\frac{\partial T(0,t)}{\partial x}=0$$

$$T(L,t)=60°\text{C}$$

FIGURE 2–29

A plane wall with insulation and specified temperature boundary conditions.

Special Case 2: Thermal Symmetry

Two surfaces of thickness *L* exposed to the same thermal conditions, and thus the temperature distribution in one half of the plate is the same as that in the other half.

Thus, the heat transfer in this plate possesses thermal symmetry about the center plane at $x = L/2$.

Therefore, the center plane can be viewed as an **insulated surface**, and the thermal condition at this plane of symmetry can be expressed as

$$\frac{\partial T(L/2,\, t)}{\partial x} = 0$$

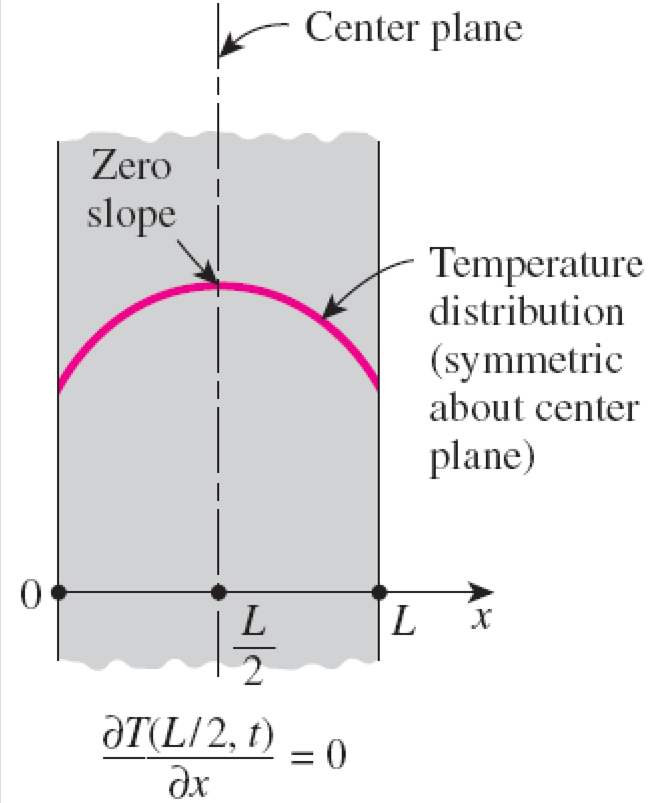


FIGURE 2–30

Thermal symmetry boundary condition at the center plane of a plane wall.

Boundary Conditions

- Specified Temperature Boundary Condition
- Specified Heat Flux Boundary Condition
- Convection Boundary Condition
- Radiation Boundary Condition
- Interface Boundary Conditions
- Generalized Boundary Conditions

3 Convection Boundary Condition

The convection boundary conditions on both surfaces in the figure can be expressed as:

$$\begin{pmatrix}\text{Heat conduction}\\ \text{at the surface in a}\\ \text{selected direction}\end{pmatrix} = \begin{pmatrix}\text{Heat convection}\\ \text{at the surface in}\\ \text{the same direction}\end{pmatrix}$$

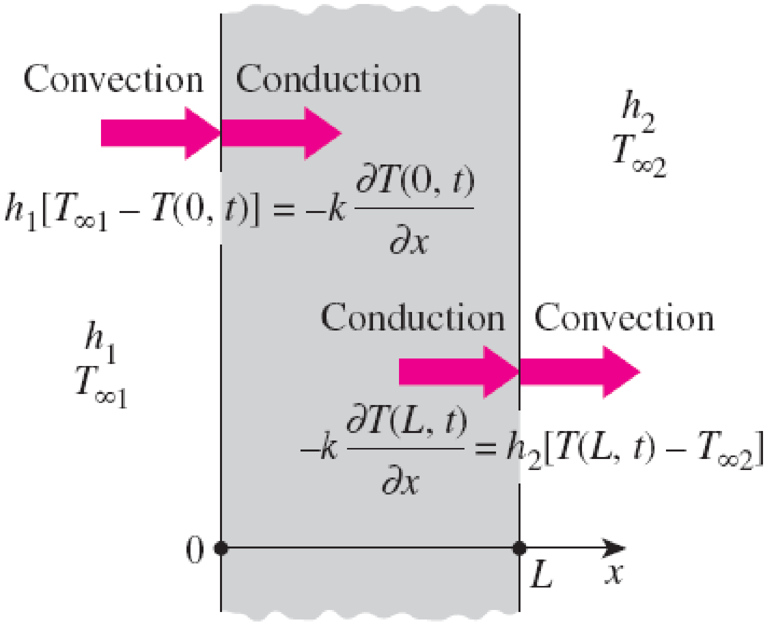


FIGURE 2–32

Convection boundary conditions on the two surfaces of a plane wall.

$$-k \frac{\partial T(0, t)}{\partial x} = h_1[T_{\infty 1} - T(0, t)]$$

$$-k \frac{\partial T(L, t)}{\partial x} = h_2[T(L, t) - T_{\infty 2}]$$

Boundary Conditions

- Specified Temperature Boundary Condition
- Specified Heat Flux Boundary Condition
- Convection Boundary Condition
- Radiation Boundary Condition
- Interface Boundary Conditions
- Generalized Boundary Conditions

4 Radiation Boundary Condition

Radiation boundary condition on a surface:

$$\begin{pmatrix}\text{Heat conduction}\\ \text{at the surface in a}\\ \text{selected direction}\end{pmatrix} = \begin{pmatrix}\text{Radiation exchange}\\ \text{at the surface in}\\ \text{the same direction}\end{pmatrix}$$

The radiation boundary conditions in the figure can be expressed as

$$-k\frac{\partial T(0,t)}{\partial x} = \varepsilon_1\sigma[T^4_{\text{surr},1} - \mathrm{T}(0,t)^4]$$

$$-k\frac{\partial T(L,t)}{\partial x} = \varepsilon_2\sigma[T(L,t)^4 - T^4_{\text{surr},2}]$$

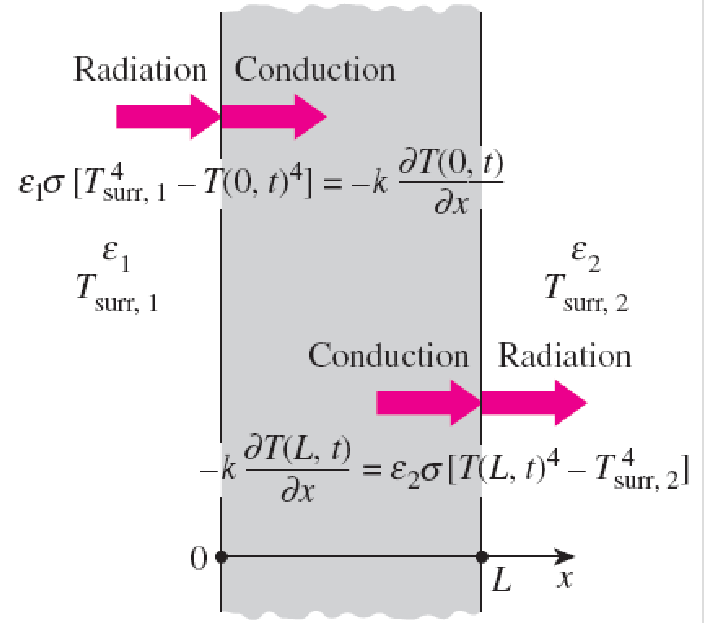


FIGURE 2–35

Radiation boundary conditions on both surfaces of a plane wall.

Boundary Conditions

- Specified Temperature Boundary Condition
- Specified Heat Flux Boundary Condition
- Convection Boundary Condition
- Radiation Boundary Condition
- Interface Boundary Conditions
- Generalized Boundary Conditions

5 Interface Boundary Conditions

Interface boundary conditions are based on

(1) two bodies must have the *same temperature* at the area of contact

(2) an interface surface cannot store any energy, and thus the *heat flux* on the two sides of an interface *must be the same.*

The boundary conditions at the interface of two bodies *A* and *B* in perfect contact at $x = x_0$ can be expressed as

$$T_A(x_0, t) = T_B(x_0, t)$$

$$-k_A \frac{\partial T_A(x_0, t)}{\partial x} = -k_B \frac{\partial T_B(x_0, t)}{\partial x}$$

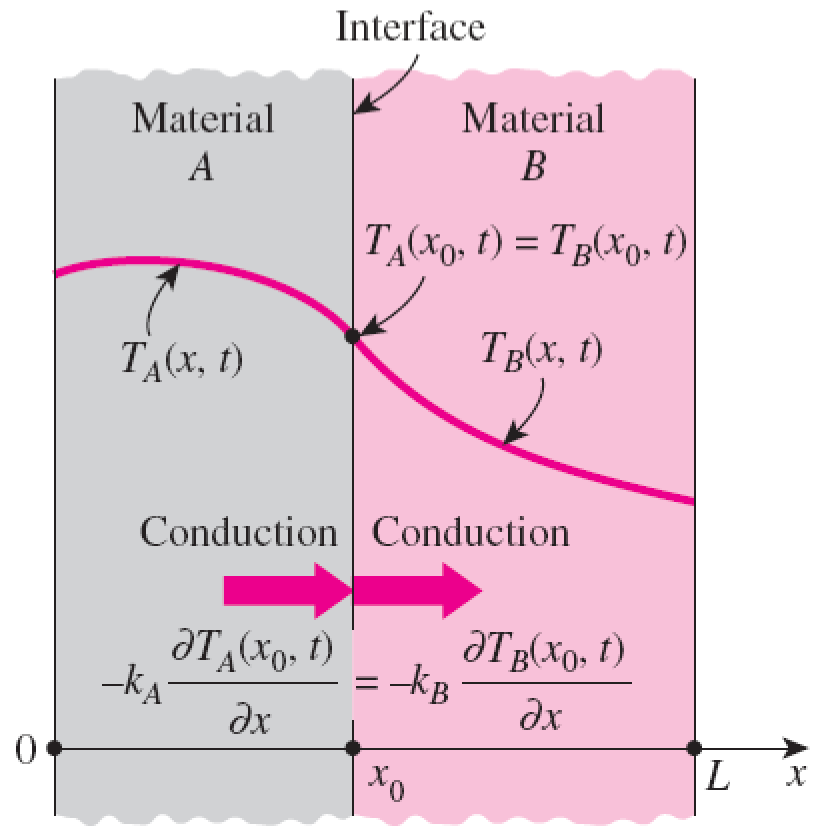


FIGURE 2–36

Boundary conditions at the interface of two bodies in perfect contact.

Boundary Conditions

- Specified Temperature Boundary Condition
- Specified Heat Flux Boundary Condition
- Convection Boundary Condition
- Radiation Boundary Condition
- Interface Boundary Conditions
- Generalized Boundary Conditions

6 Generalized Boundary Conditions

In general, however, a surface may involve convection, radiation, *and* specified heat flux simultaneously.

The boundary condition in such cases is again obtained from a surface energy balance, expressed as

$$\begin{pmatrix} \text{Heat transfer} \\ \text{to the surface} \\ \text{in all modes} \end{pmatrix} = \begin{pmatrix} \text{Heat transfer} \\ \text{from the surface} \\ \text{in all modes} \end{pmatrix}$$

EXAMPLE 2–8 **Combined Convection and Radiation Condition**

A spherical metal ball of radius r_o is heated in an oven to a temperature of 300°C throughout and is then taken out of the oven and allowed to cool in ambient air at $T_\infty = 27°C$, as shown in Fig. 2–37. The thermal conductivity of the ball material is $k = 14.4$ W/m·K, and the average convection heat transfer coefficient on the outer surface of the ball is evaluated to be $h = 25$ W/m²·K. The emissivity of the outer surface of the ball is $\varepsilon = 0.6$, and the average temperature of the surrounding surfaces is $T_{surr} = 290$ K. Assuming the ball is cooled uniformly from the entire outer surface, express the initial and boundary conditions for the cooling process of the ball.

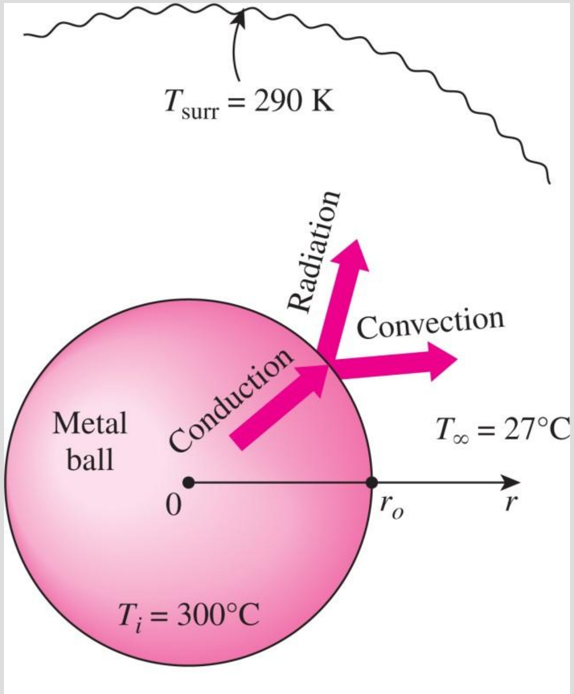


FIGURE 2–37

Schematic for Example 2–8.

SOLUTION The cooling of a hot spherical metal ball is considered. The initial and boundary conditions are to be obtained.

Analysis The ball is initially at a uniform temperature and is cooled uniformly from the entire outer surface. Therefore, this is a one-dimensional transient heat transfer problem since the temperature within the ball changes with the radial distance r and the time t. That is, $T = T(r, t)$. Taking the moment the ball is removed from the oven to be $t = 0$, the initial condition can be expressed as

$$T(r, 0) = T_i = 300°C$$

The problem possesses symmetry about the midpoint ($r = 0$) since the isotherms in this case are concentric spheres, and thus no heat is crossing the midpoint of the ball. Then the boundary condition at the midpoint can be expressed as

$$\frac{\partial T(0, t)}{\partial r} = 0$$

The heat conducted to the outer surface of the ball is lost to the environment by convection and radiation. Then taking the direction of heat transfer to be the positive r direction, the boundary condition on the outer surface can be expressed as

$$-k\frac{\partial T(r_o, t)}{\partial r} = h[T(r_o) - T_\infty] + \varepsilon\sigma[T(r_o)^4 - T_{surr}^4]$$

Discussion All the quantities in the above relations are known except the temperatures and their derivatives at $r = 0$ and r_o. Also, the radiation part of the boundary condition is often ignored for simplicity by modifying the convection heat transfer coefficient to account for the contribution of radiation. The convection coefficient h in that case becomes the combined heat transfer coefficient.

EXAMPLE 2–9 Combined Convection, Radiation, and Heat Flux

Consider the south wall of a house that is $L = 0.2$ m thick. The outer surface of the wall is exposed to solar radiation and has an absorptivity of $\alpha = 0.5$ for solar energy. The interior of the house is maintained at $T_{\infty 1} = 20°C$, while the ambient air temperature outside remains at $T_{\infty 2} = 5°C$. The sky, the ground, and the surfaces of the surrounding structures at this location can be modeled as a surface at an effective temperature of $T_{sky} = 255$ K for radiation exchange on the outer surface. The radiation exchange between the inner surface of the wall and the surfaces of the walls, floor, and ceiling it faces is negligible. The convection heat transfer coefficients on the inner and the outer surfaces of the wall are $h_1 = 6$ W/m²·K and $h_2 = 25$ W/m²·K, respectively. The thermal conductivity of the wall material is $k = 0.7$ W/m·K, and the emissivity of the outer surface is $\varepsilon_2 = 0.9$. Assuming the heat transfer through the wall to be steady and one-dimensional, express the boundary conditions on the inner and the outer surfaces of the wall.

SOLUTION The wall of a house subjected to solar radiation is considered. The boundary conditions on the inner and outer surfaces of the wall are to be obtained.

Analysis We take the direction normal to the wall surfaces as the x-axis with the origin at the inner surface of the wall, as shown in Fig. 2–38. The heat transfer through the wall is given to be steady and one-dimensional, and thus the temperature depends on x only and not on time. That is, $T = T(x)$.

The boundary condition on the inner surface of the wall at $x = 0$ is a typical convection condition since it does not involve any radiation or specified heat flux. Taking the direction of heat transfer to be the positive x-direction, the boundary condition on the inner surface can be expressed as

$$-k\frac{dT(0)}{dx} = h_1[T_{\infty 1} - T(0)]$$

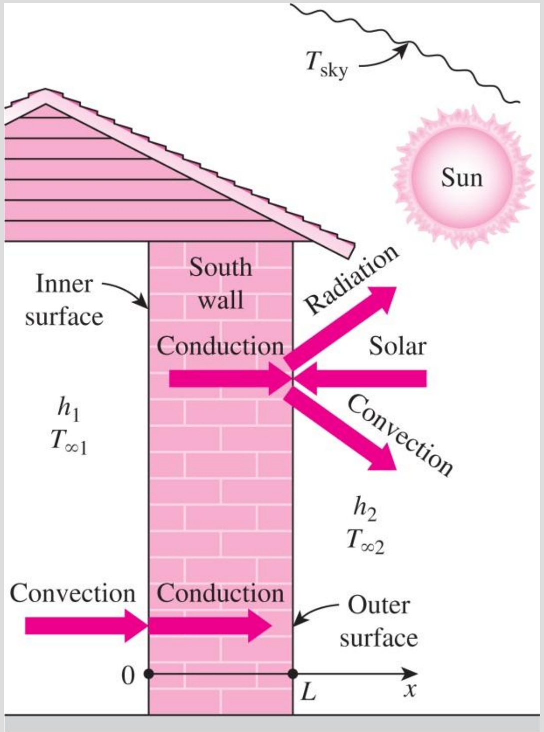


FIGURE 2–38
Schematic for Example 2–9.

The boundary condition on the outer surface at $x = 0$ is quite general as it involves conduction, convection, radiation, and specified heat flux. Again taking the direction of heat transfer to be the positive x-direction, the boundary condition on the outer surface can be expressed as

$$-k\frac{dT(L)}{dx} = h_2[T(L) - T_{\infty 2}] + \varepsilon_2\sigma[T(L)^4 - T_{sky}^4] - \alpha\dot{q}_{solar}$$

where $\dot{q}_{solar}$ is the incident solar heat flux.

Discussion Assuming the opposite direction for heat transfer would give the same result multiplied by -1, which is equivalent to the relation here. All the quantities in these relations are known except the temperatures and their derivatives at the two boundaries.

EXAMPLE 2–10 Heat Conduction in a Plane Wall

Consider a large plane wall of thickness $L = 0.2$ m, thermal conductivity $k = 1.2$ W/m·K, and surface area $A = 15$ m^2. The two sides of the wall are maintained at constant temperatures of $T_1 = 120°C$ and $T_2 = 50°C$, respectively, as shown in Fig. 2–40. Determine (*a*) the variation of temperature within the wall and the value of temperature at $x = 0.1$ m and (*b*) the rate of heat conduction through the wall under steady conditions.

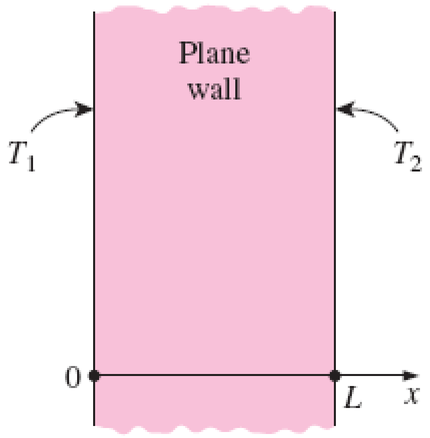


FIGURE 2–40

Schematic for Example 2–10.

SOLUTION A plane wall with specified surface temperatures is given. The variation of temperature and the rate of heat transfer are to be determined.

Assumptions **1** Heat conduction is steady. **2** Heat conduction is one-dimensional since the wall is large relative to its thickness and the thermal conditions on both sides are uniform. **3** Thermal conductivity is constant. **4** There is no heat generation.

Properties The thermal conductivity is given to be $k = 1.2$ W/m·K.

Analysis (*a*) Taking the direction normal to the surface of the wall to be the *x*-direction, the differential equation for this problem can be expressed as

$$\frac{d^2T}{dx^2} = 0$$

with boundary conditions

$$T(0) = T_1 = 120°C$$
$$T(L) = T_2 = 50°C$$

The differential equation is linear and second order, and a quick inspection of it reveals that it has a single term involving derivatives and no terms involving the unknown function T as a factor. Thus, it can be solved by direct integration. Noting that an integration reduces the order of a derivative by one, the general solution of the differential equation above can be obtained by two simple successive integrations, each of which introduces an integration constant.

Integrating the differential equation once with respect to x yields

$$\frac{dT}{dx} = C_1$$

where C_1 is an arbitrary constant. Notice that the order of the derivative went down by one as a result of integration. As a check, if we take the derivative of this equation, we will obtain the original differential equation. This equation is not the solution yet since it involves a derivative.

Integrating one more time, we obtain

$$T(x) = C_1x + C_2$$

which is the general solution of the differential equation (Fig. 2–41). The general solution in this case resembles the general formula of a straight line whose slope is C_1 and whose value at $x = 0$ is C_2. This is not surprising since the second derivative represents the change in the slope of a function, and a zero second derivative indicates that the slope of the function remains constant. Therefore, *any straight line* is a solution of this differential equation.

The general solution contains two unknown constants C_1 and C_2, and thus we need two equations to determine them uniquely and obtain the specific solution. These equations are obtained by forcing the general solution to satisfy the specified boundary conditions. The application of each condition yields one equation, and thus we need to specify two conditions to determine the constants C_1 and C_2.

When applying a boundary condition to an equation, *all occurrences of the dependent and independent variables and any derivatives are replaced by the specified values.* Thus the only unknowns in the resulting equations are the arbitrary constants.

The first boundary condition can be interpreted as *in the general solution, replace all the x's by zero and T(x) by T_1*. That is (Fig. 2–42),

$$T(0) = C_1 \times 0 + C_2 \quad \rightarrow \quad C_2 = T_1$$

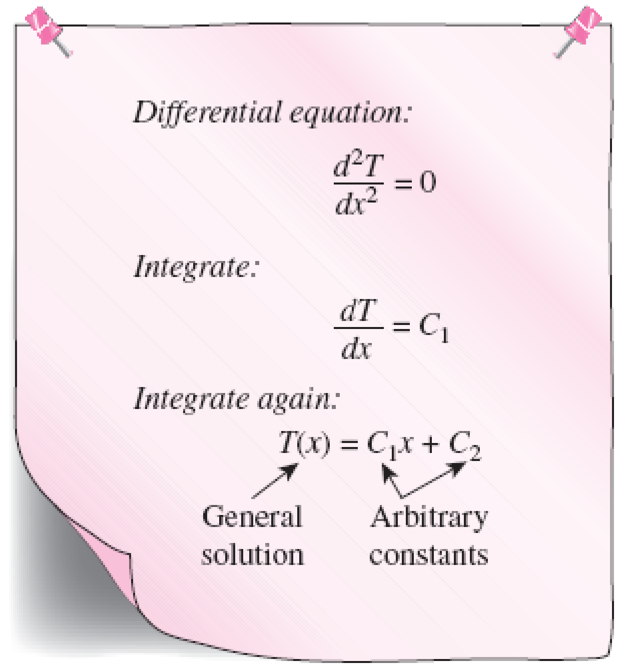


FIGURE 2–41
Obtaining the general solution of a simple second order differential equation by integration.

The second boundary condition can be interpreted as *in the general solution, replace all the x's by L and T(x) by T_2*. That is,

$$T(L) = C_1L + C_2 \quad \rightarrow \quad T_2 = C_1L + T_1 \quad \rightarrow \quad C_1 = \frac{T_2 - T_1}{L}$$

Substituting the C_1 and C_2 expressions into the general solution, we obtain

$$T(x) = \frac{T_2 - T_1}{L}x + T_1 \tag{2–56}$$

which is the desired solution since it satisfies not only the differential equation but also the two specified boundary conditions. That is, differentiating Eq. 2–56 with respect to x twice will give d^2T/dx^2, which is the given differential equation, and substituting $x = 0$ and $x = L$ into Eq. 2–56 gives $T(0) = T_1$ and $T(L) = T_2$, respectively, which are the specified conditions at the boundaries.

Substituting the given information, the value of the temperature at $x = 0.1$ m is determined to be

$$T(0.1 \text{ m}) = \frac{(50 - 120)°\text{C}}{0.2 \text{ m}}(0.1 \text{ m}) + 120°\text{C} = 85°\text{C}$$

(*b*) The rate of heat conduction anywhere in the wall is determined from Fourier's law to be

$$\dot{Q}_{\text{wall}} = -kA\frac{dT}{dx} = -kAC_1 = -kA\frac{T_2 - T_1}{L} = kA\frac{T_1 - T_2}{L} \tag{2–57}$$

The numerical value of the rate of heat conduction through the wall is determined by substituting the given values to be

$$\dot{Q} = kA\frac{T_1 - T_2}{L} = (1.2 \text{ W/m·K})(15 \text{ m}^2)\frac{(120 - 50)°\text{C}}{0.2 \text{ m}} = 6300 \text{ W}$$

Discussion Note that under steady conditions, the rate of heat conduction through a plane wall is constant.

Boundary condition:

$$T(0) = T_1$$

General solution:

$$T(x) = C_1x + C_2$$

Applying the boundary condition:

$$\underbrace{\underset{\overset{\uparrow}{0}}{T(x)}}_{T_1} = C_1 \underset{\overset{\uparrow}{0}}{x} + C_2$$

Substituting:

$$T_1 = C_1 \times 0 + C_2 \rightarrow C_2 = T_1$$

It cannot involve x or $T(x)$ after the boundary condition is applied.

FIGURE 2–42

When applying a boundary condition to the general solution at a specified point, all occurrences of the dependent and independent variables should be replaced by their specified values at that point.