EXAMPLE 5.4

At low velocities (laminar flow), the volume flow Q through a small-bore tube is a function only of the tube radius R, the fluid viscosity μ, and the pressure drop per unit tube length dp/dx. Using the pi theorem, find an appropriate dimensionless relationship.

Solution

Write the given relation and count variables:

$$Q = f\left(R, \mu, \frac{dp}{dx}\right) \quad \text{four variables } (n = 4)$$

Make a list of the dimensions of these variables from Table 5.1 using the $\{MLT\}$ system:

Q	R	μ	dp/dx
$\{L^3T^{-1}\}$	$\{L\}$	$\{ML^{-1}T^{-1}\}$	$\{ML^{-2}T^{-2}\}$

There are three primary dimensions (M, L, T), hence $j \le 3$. By trial and error we determine that R, μ, and dp/dx cannot be combined into a pi group. Then $j = 3$, and $n - j = 4 - 3 = 1$. There is only *one* pi group, which we find by combining Q in a power product with the other three:

$$\Pi_1 = R^a \mu^b \left(\frac{dp}{dx}\right)^c Q^1 = (L)^a (ML^{-1}T^{-1})^b (ML^{-2}T^{-2})^c (L^3T^{-1})$$
$$= M^0 L^0 T^0$$

Equate exponents:

Mass: $\quad b + c = 0$

Length: $\quad a - b - 2c + 3 = 0$

Time: $\quad -b - 2c - 1 = 0$

Solving simultaneously, we obtain $a = -4$, $b = 1$, and $c = -1$. Then

$$\Pi_1 = R^{-4} \mu^1 \left(\frac{dp}{dx}\right)^{-1} Q$$

or

$$\Pi_1 = \frac{Q\mu}{R^4(dp/dx)} = \text{const} \qquad \textit{Ans.}$$

Since there is only one pi group, it must equal a dimensionless constant. This is as far as dimensional analysis can take us. The laminar flow theory of Sec. 4.10 shows that the value of the constant is $-\frac{\pi}{8}$.

EXAMPLE 5.5

Assume that the tip deflection δ of a cantilever beam is a function of the tip load P, beam length L, area moment of inertia I, and material modulus of elasticity E; that is, $\delta = f(P, L, I, E)$. Rewrite this function in dimensionless form, and comment on its complexity and the peculiar value of j.

Solution

List the variables and their dimensions:

δ	P	L	I	E
$\{L\}$	$\{MLT^{-2}\}$	$\{L\}$	$\{L^4\}$	$\{ML^{-1}T^{-2}\}$

There are five variables ($n = 5$) and three primary dimensions (M, L, T), hence $j \leq 3$. But try as we may, we *cannot* find any combination of three variables that does not form a pi group. This is because $\{M\}$ and $\{T\}$ occur only in P and E and only in the same form, $\{MT^{-2}\}$. Thus we have encountered a special case of $j = 2$, which is less than the number of dimensions (M, L, T). To gain more insight into this peculiarity, you should rework the problem, using the (F, L, T) system of dimensions. You will find that only $\{F\}$ and $\{L\}$ occur in these variables, hence $j = 2$.

With $j = 2$, we select L and E as two variables that cannot form a pi group and then add other variables to form the three desired pis:

$$\Pi_1 = L^a E^b I^1 = (L)^a (ML^{-1}T^{-2})^b (L^4) = M^0 L^0 T^0$$

from which, after equating exponents, we find that $a = -4$, $b = 0$, or $\Pi_1 = I/L^4$. Then

$$\Pi_2 = L^a E^b P^1 = (L)^a (ML^{-1}T^{-2})^b (MLT^{-2}) = M^0 L^0 T^0$$

from which we find $a = -2$, $b = -1$, or $\Pi_2 = P/(EL^2)$, and

$$\Pi_3 = L^a E^b \delta^1 = (L)^a (ML^{-1}T^{-2})^b (L) = M^0 L^0 T^0$$

EXAMPLE 5.2

Repeat the development of Eq. (5.2) from Eq. (5.1), using the pi theorem.

Solution

Step 1 Write the function and count variables:

$$F = f(L, U, \rho, \mu) \quad \text{there are five variables } (n = 5)$$

Step 2 List dimensions of each variable. From Table 5.1

F	L	U	ρ	μ
$\{MLT^{-2}\}$	$\{L\}$	$\{LT^{-1}\}$	$\{ML^{-3}\}$	$\{ML^{-1}T^{-1}\}$

Step 3 Find j. No variable contains the dimension Θ, and so j is less than or equal to 3 (MLT). We inspect the list and see that L, U, and ρ cannot form a pi group because only ρ contains mass and only U contains time. Therefore j does equal 3, and $n - j = 5 - 3 = 2 = k$. The pi theorem guarantees for this problem that there will be exactly two independent dimensionless groups.

Step 4 Select repeating j variables. The group L, U, ρ we found in step 3 will do fine.

Step 5 Combine L, U, ρ with one additional variable, in sequence, to find the two pi products.

First add force to find Π_1. You may select *any* exponent on this additional term as you please, to place it in the numerator or denominator to any power. Since F is the output, or dependent, variable, we select it to appear to the first power in the numerator:

$$\Pi_1 = L^a U^b \rho^c F = (L)^a (LT^{-1})^b (ML^{-3})^c (MLT^{-2}) = M^0 L^0 T^0$$

Equate exponents:

Length: $$a + b - 3c + 1 = 0$$

Mass: $$c + 1 = 0$$

Time: $$-b \qquad -2 = 0$$

We can solve explicitly for

$$a = -2 \qquad b = -2 \qquad c = -1$$

Therefore

$$\Pi_1 = L^{-2} U^{-2} \rho^{-1} F = \frac{F}{\rho U^2 L^2} = C_F \qquad \textit{Ans.}$$

This is exactly the right pi group as in Eq. (5.2). By varying the exponent on F, we could have found other equivalent groups such as $UL\rho^{1/2}/F^{1/2}$.

Finally, add viscosity to L, U, and ρ to find Π_2. Select any power you like for viscosity. By hindsight and custom, we select the power -1 to place it in the denominator:

$$\Pi_2 = L^a U^b \rho^c \mu^{-1} = L^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^{-1} = M^0 L^0 T^0$$

Equate exponents:

Length: $$a + b - 3c + 1 = 0$$

Mass: $$c - 1 = 0$$

Time: $$-b \qquad + 1 = 0$$

from which we find

$$a = b = c = 1$$

Therefore $$\Pi_2 = L^1 U^1 \rho^1 \mu^{-1} = \frac{\rho U L}{\mu} = \text{Re}$$ *Ans.*

We know we are finished; this is the second and last pi group. The theorem guarantees that the functional relationship must be of the equivalent form

$$\frac{F}{\rho U^2 L^2} = g\left(\frac{\rho U L}{\mu}\right)$$ *Ans.*

which is exactly Eq. (5.2).

EXAMPLE 5.3

Reduce the falling-body relationship, Eq. (5.5), to a function of dimensionless variables. Why are there three different formulations?

Solution

Write the function and count variables

$$S = f(t, S_0, V_0, g) \quad \text{five variables } (n = 5)$$

List the dimensions of each variable, from Table 5.1:

S	t	S_0	V_0	g
$\{L\}$	$\{T\}$	$\{L\}$	$LT^{-1}\}$	$\{LT^{-2}\}$

There are only two primary dimensions (L, T), so that $j \le 2$. By inspection we can easily find two variables which cannot be combined to form a pi, for example, V_0 and g. Then $j = 2$, and we expect $5 - 2 = 3$ pi products. Select j variables among the parameters S_0, V_0, and g. Avoid S and t since they are the dependent variables, which should not be repeated in pi groups.

There are three different options for repeating variables among the group (S_0, V_0, g). Therefore we can obtain three different dimensionless formulations, just as we did informally with the falling-body equation in Sec. 5.2. Take each option in turn:

1. Choose S_0 and V_0 as repeating variables. Combine them in turn with (S, t, g):

$$\Pi_1 = S^1 S_0^a V_0^b \quad \Pi_2 = t^1 S_0^c V_0^d \quad \Pi_3 = g^1 S_0^e V_0^f$$

Set each power product equal to L^0T^0, and solve for the exponents (a, b, c, d, e, f). Please allow us to give the results here, and you may check the algebra as an exercise:

$$a = -1 \quad b = 0 \quad c = -1 \quad d = 1 \quad e = 1 \quad f = -2$$

$$\Pi_1 = S^* = \frac{S}{S_0} \qquad \Pi_2 = t^* = \frac{V_0 t}{S_0} \qquad \Pi_3 = \alpha = \frac{gS_0}{V_0^2} \qquad \textit{Ans.}$$

Thus, for option 1, we know that $S^* = \text{fcn}(t^*, \alpha)$. We have found, by dimensional analysis, the same variables as in Eq. (5.10). But here there is no *formula* for the functional relation — we might have to experiment with falling bodies to establish Fig. 5.1*a*.

2. Choose V_0 and g as repeating variables. Combine them in turn with (S, t, S_0):

$$\Pi_1 = S^1 V_0^a g^b \quad \Pi_2 = t^1 V_0^c g^d \quad \Pi_3 = S_0^1 V_0^e g^f$$

Set each power product equal to L^0T^0, and solve for the exponents (a, b, c, d, e, f). Once more allow us to give the results here, and you may check the algebra as an exercise.

$$a = -2 \quad b = 1 \quad c = -1 \quad d = 1 \quad e = 1 \quad f = -2$$

$$\Pi_1 = S^{**} = \frac{Sg}{V_0^2} \quad \Pi_2 = t^{**} = \frac{tg}{V_0} \quad \Pi_3 = \alpha = \frac{gS_0}{V_0^2} \qquad \textit{Ans.}$$

Thus, for option 2, we now know that $S^{**} = \text{fcn}(t^{**}, \alpha)$. We have found, by dimensional analysis, the same groups as in Eq. (5.12). The data would plot as in Fig. 5.1*b*.

3. Finally choose S_0 and g as repeating variables. Combine them in turn with (S, t, V_0):

$$\Pi_1 = S^1 S_0^a g^b \quad \Pi_2 = t^1 S_0^c g^d \quad \Pi_3 = V_0^1 S_0^e g^f$$

Set each power product equal to L^0T^0, and solve for the exponents (a, b, c, d, e, f). One more time allow us to give the results here, and you may check the algebra as an exercise:

$$a = -1 \quad b = 0 \quad c = -\frac{1}{2} \quad d = \frac{1}{2} \quad e = -\frac{1}{2} \quad f = -\frac{1}{2}$$

$$\Pi_1 = S^{***} = \frac{S}{S_0} \quad \Pi_2 = t^{***} = t\sqrt{\frac{g}{S_0}} \quad \Pi_3 = \beta = \frac{V_0}{\sqrt{gS_0}} \qquad \textit{Ans.}$$

Thus, for option 3, we now know that $S^{***} = \text{fcn}(t^{***}, \beta = 1/\sqrt{\alpha})$. We have found, by dimensional analysis, the same groups as in Eq. (5.14). The data would plot as in Fig. 5.1*c*.

Dimensional analysis here has yielded the same pi groups as the use of scaling parameters with Eq. (5.5). Three different formulations appeared, because we could choose three different pairs of repeating variables to complete the pi theorem.

EXAMPLE 5.4

At low velocities (laminar flow), the volume flow Q through a small-bore tube is a function only of the tube radius R, the fluid viscosity μ, and the pressure drop per unit tube length dp/dx. Using the pi theorem, find an appropriate dimensionless relationship.

Solution

Write the given relation and count variables:

$$Q = f\left(R, \mu, \frac{dp}{dx}\right) \quad \text{four variables } (n = 4)$$

Make a list of the dimensions of these variables from Table 5.1:

Q	R	μ	dp/dx
$\{L^3T^{-1}\}$	$\{L\}$	$\{ML^{-1}T^{-1}\}$	$\{ML^{-2}T^{-2}\}$

There are three primary dimensions (M, L, T), hence $j \le 3$. By trial and error we determine that R, μ, and dp/dx cannot be combined into a pi group. Then $j = 3$, and $n - j = 4 - 3 = 1$. There is only *one* pi group, which we find by combining Q in a power product with the other three:

$$\Pi_1 = R^a \mu^b \left(\frac{dp}{dx}\right)^c Q^1 = (L)^a (ML^{-1}T^{-1})^b (ML^{-2}T^{-2})^c (L^3T^{-1})$$

$$= M^0 L^0 T^0$$

Equate exponents:

Mass: $\quad b + c = 0$

Length: $\quad a - b - 2c + 3 = 0$

Time: $\quad -b - 2c - 1 = 0$

Solving simultaneously, we obtain $a = -4$, $b = 1$, $c = -1$. Then

$$\Pi_1 = R^{-4}\mu^1 \left(\frac{dp}{dx}\right)^{-1} Q$$

or

$$\Pi_1 = \frac{Q\mu}{R^4(dp/dx)} = \text{const} \qquad \textit{Ans.}$$

Since there is only one pi group, it must equal a dimensionless constant. This is as far as dimensional analysis can take us. The laminar-flow theory of Sec. 6.4 shows that the value of the constant is $\pi/8$.

EXAMPLE 5.5

Assume that the tip deflection δ of a cantilever beam is a function of the tip load P, beam length L, area moment of inertia I, and material modulus of elasticity E; that is, $\delta = f(P, L, I, E)$. Rewrite this function in dimensionless form, and comment on its complexity and the peculiar value of j.

Solution

List the variables and their dimensions:

δ	P	L	I	E
$\{L\}$	$\{MLT^{-2}\}$	$\{L\}$	$\{L^4\}$	$\{ML^{-1}T^{-2}\}$

There are five variables ($n = 5$) and three primary dimensions (M, L, T), hence $j \leq 3$. But try as we may, we *cannot* find any combination of three variables which does not form a pi group. This is because $\{M\}$ and $\{T\}$ occur only in P and E and only in the same form, $\{MT^{-2}\}$. Thus we have encountered a special case of $j = 2$, which is less than the number of dimensions (M, L, T). To gain more insight into this peculiarity, you should rework the problem, using the (F, L, T) system of dimensions.

With $j = 2$, we select L and E as two variables which cannot form a pi group and then add other variables to form the three desired pis:

$$\Pi_1 = L^a E^b I^1 = (L)^a (ML^{-1}T^{-2})^b (L^4) = M^0 L^0 T^0$$

from which, after equating exponents, we find that $a = -4$, $b = 0$, or $\Pi_1 = I/L^4$. Then

$$\Pi_2 = L^a E^b P^1 = (L)^a (ML^{-1}T^{-2})^b (MLT^{-2}) = M^0 L^0 T^0$$

from which we find $a = -2$, $b = -1$, or $\Pi_2 = P/(EL^2)$, and

$$\Pi_3 = L^a E^b \delta^1 = (L)^a (ML^{-1}T^{-2})^b (L) = M^0 L^0 T^0$$

from which $a = -1$, $b = 0$, or $\Pi_3 = \delta/L$. The proper dimensionless function is $\Pi_3 = f(\Pi_2, \Pi_1)$, or

$$\frac{\delta}{L} = f\left(\frac{P}{EL^2}, \frac{I}{L^4}\right) \qquad \textit{Ans.} \quad (1)$$

This is a complex three-variable function, but dimensional analysis alone can take us no further.

We can "improve" Eq. (1) by taking advantage of some physical reasoning, as Langhaar points out [8, p. 91]. For small elastic deflections, δ is proportional to load P and inversely proportional to moment of inertia I. Since P and I occur separately in Eq. (1), this means that Π_3 must be proportional to Π_2 and inversely proportional to Π_1. Thus, for these conditions,

$$\frac{\delta}{L} = (\text{const}) \frac{P}{EL^2} \frac{L^4}{I}$$

or

$$\delta = (\text{const}) \frac{PL^3}{EI} \qquad (2)$$

This could not be predicted by a pure dimensional analysis. Strength-of-materials theory predicts that the value of the constant is $\frac{1}{3}$.

EXAMPLE 5.6

The capillary rise h of a liquid in a tube varies with tube diameter d, gravity g, fluid density ρ, surface tension Υ, and the contact angle θ. (*a*) Find a dimensionless statement of this relation. (*b*) If $h = 3$ cm in a given experiment, what will h be in a similar case if the diameter and surface tension are half as much, the density is twice as much, and the contact angle is the same?

Solution

Part (a) Step 1 Write down the function and count variables:

$$h = f(d, g, \rho, \Upsilon, \theta) \qquad n = 6 \text{ variables}$$

Step 2 List the dimensions $\{FLT\}$ from Table 5.2:

h	d	g	ρ	Υ	θ
$\{L\}$	$\{L\}$	$\{LT^{-2}\}$	$\{FT^2L^{-4}\}$	$\{FL^{-1}\}$	none

Step 3 Find j. Several groups of three form no pi: Υ, ρ, and g or ρ, g, and d. Therefore $j = 3$, and we expect $n - j = 6 - 3 = 3$ dimensionless groups. One of these is obviously θ, which is already dimensionless:

$$\Pi_3 = \theta \qquad \textit{Ans. (a)}$$

If we had carelessly chosen to search for it by using steps 4 and 5, we would still find $\Pi_3 = \theta$.

Step 4 Select j repeating variables which do not form a pi group: ρ, g, d.

Step 5 Add one additional variable in sequence to form the pis:

Add h: $$\Pi_1 = \rho^a g^b d^c h = (FT^2L^{-4})^a(LT^{-2})^b(L)^c(L) = F^0L^0T^0$$

Solve for

$$a = b = 0 \qquad c = -1$$

Therefore $$\Pi_1 = \rho^0 g^0 d^{-1} h = \frac{h}{d} \qquad \textit{Ans. (a)}$$

Finally add Υ, again selecting its exponent to be 1:

$$\Pi_2 = \rho^a g^b d^c \Upsilon = (FT^2L^{-4})^a(LT^{-2})^b(L)^c(FL^{-1}) = F^0L^0T^0$$

Solve for

$$a = b = -1 \qquad c = -2$$

Therefore $$\Pi_2 = \rho^{-1} g^{-1} d^{-2} \Upsilon = \frac{\Upsilon}{\rho g d^2} \qquad \textit{Ans. (a)}$$

Step 6 The complete dimensionless relation for this problem is thus

$$\frac{h}{d} = F\left(\frac{\Upsilon}{\rho g d^2}, \theta\right) \qquad \textit{Ans. (a)} \quad (1)$$

This is as far as dimensional analysis goes. Theory, however, establishes that h is proportional to Υ. Since Υ occurs only in the second parameter, we can slip it outside

$$\left(\frac{h}{d}\right)_{\text{actual}} = \frac{\Upsilon}{\rho g d^2} F_1(\theta) \qquad \text{or} \qquad \frac{h\rho g d}{\Upsilon} = F_1(\theta)$$

Example 1.9 showed theoretically that $F_1(\theta) = 4 \cos \theta$.

Part (b) We are given h_1 for certain conditions d_1, Υ_1, ρ_1, and θ_1. If $h_1 = 3$ cm, what is h_2 for $d_2 = \frac{1}{2}d_1$, $\Upsilon_2 = \frac{1}{2}\Upsilon_1$, $\rho_2 = 2\rho_1$, and $\theta_2 = \theta_1$? We know the functional relation, Eq. (1), must still hold at condition 2

$$\frac{h_2}{d_2} = F\left(\frac{\Upsilon_2}{\rho_2 g d_2^2}, \theta_2\right)$$

But

$$\frac{\Upsilon_2}{\rho_2 g d_2^2} = \frac{\frac{1}{2}\Upsilon_1}{2\rho_1 g(\frac{1}{2}d_1)^2} = \frac{\Upsilon_1}{\rho_1 g d_1^2}$$

EXAMPLE 5.3

The power input P to a centrifugal pump is a function of the volume flow Q, impeller diameter D, rotational rate Ω, and the density ρ and viscosity μ of the fluid:

$$P = f(Q, D, \Omega, \rho, \mu)$$

Rewrite this as a dimensionless relationship. *Hint:* Use Ω, ρ, and D as repeating variables.