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Electromagnetic Fields Theory

For

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Second Semester

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CHAPTER 1

Vector Analysis

Vector analysis is a mathematical subject that is better taught by mathematicians than by engineers. Most junior and senior engineering students have not had the time (or the inclination) to take a course in vector analysis, although it is likely that vector concepts and operations were introduced in the calculus sequence. These are covered in this chapter, and the time devoted to them now should depend on past exposure.

The viewpoint here is that of the engineer or physicist and not that of the mathematician. Proofs are indicated rather than rigorously expounded, and physical interpretation is stressed. It is easier for engineers to take a more rigorous course in the mathematics department after they have been presented with a few physical pictures and applications.

Vector analysis is a mathematical shorthand. It has some new symbols and some new rules, and it demands concentration and practice. The drill problems, first found at the end of Section 1.4, should be considered part of the text and should all be worked. They should not prove to be difficult if the material in the accompanying section of the text has been thoroughly understood. It takes a little longer to "read" the chapter this way, but the investment in time will produce a surprising interest. ■

1.1 SCALARS AND VECTORS

The term *scalar* refers to a quantity whose value may be represented by a single (positive or negative) real number. The x, y, and z we use in basic algebra are scalars, and the quantities they represent are scalars. If we speak of a body falling a distance L in a time t, or the temperature T at any point in a bowl of soup whose coordinates are x, y, and z, then L, t, T, x, y, and z are all scalars. Other scalar quantities are mass, density, pressure (but not force), volume, volume resistivity, and voltage.

A *vector* quantity has both a magnitude[1] and a direction in space. We are concerned with two- and three-dimensional spaces only, but vectors may be defined in

[1] We adopt the convention that magnitude infers absolute value; the magnitude of any quantity is, therefore, always positive.

n-dimensional space in more advanced applications. Force, velocity, acceleration, and a straight line from the positive to the negative terminal of a storage battery are examples of vectors. Each quantity is characterized by both a magnitude and a direction.

Our work will mainly concern scalar and vector *fields*. A field (scalar or vector) may be defined mathematically as some function that connects an arbitrary origin to a general point in space. We usually associate some physical effect with a field, such as the force on a compass needle in the earth's magnetic field, or the movement of smoke particles in the field defined by the vector velocity of air in some region of space. Note that the field concept invariably is related to a region. Some quantity is defined at every point in a region. Both *scalar fields* and *vector fields* exist. The temperature throughout the bowl of soup and the density at any point in the earth are examples of scalar fields. The gravitational and magnetic fields of the earth, the voltage gradient in a cable, and the temperature gradient in a soldering-iron tip are examples of vector fields. The value of a field varies in general with both position and time.

In this book, as in most others using vector notation, vectors will be indicated by boldface type, for example, **A**. Scalars are printed in italic type, for example, A. When writing longhand, it is customary to draw a line or an arrow over a vector quantity to show its vector character. (CAUTION: This is the first pitfall. Sloppy notation, such as the omission of the line or arrow symbol for a vector, is the major cause of errors in vector analysis.)

1.2 VECTOR ALGEBRA

With the definition of vectors and vector fields now established, we may proceed to define the rules of vector arithmetic, vector algebra, and (later) vector calculus. Some of the rules will be similar to those of scalar algebra, some will differ slightly, and some will be entirely new.

To begin, the addition of vectors follows the parallelogram law. Figure 1.1 shows the sum of two vectors, **A** and **B**. It is easily seen that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, or that vector addition obeys the commutative law. Vector addition also obeys the associative law,

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

Note that when a vector is drawn as an arrow of finite length, its location is defined to be at the tail end of the arrow.

Coplanar vectors are vectors lying in a common plane, such as those shown in Figure 1.1. Both lie in the plane of the paper and may be added by expressing each vector in terms of "horizontal" and "vertical" components and then adding the corresponding components.

Vectors in three dimensions may likewise be added by expressing the vectors in terms of three components and adding the corresponding components. Examples of this process of addition will be given after vector components are discussed in Section 1.4.

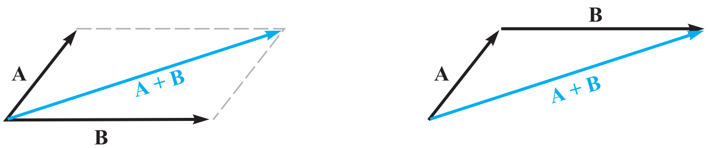

Figure 1.1 Two vectors may be added graphically either by drawing both vectors from a common origin and completing the parallelogram or by beginning the second vector from the head of the first and completing the triangle; either method is easily extended to three or more vectors.

The rule for the subtraction of vectors follows easily from that for addition, for we may always express $\mathbf{A}-\mathbf{B}$ as $\mathbf{A}+(-\mathbf{B})$; the sign, or direction, of the second vector is reversed, and this vector is then added to the first by the rule for vector addition.

Vectors may be multiplied by scalars. The magnitude of the vector changes, but its direction does not when the scalar is positive, although it reverses direction when multiplied by a negative scalar. Multiplication of a vector by a scalar also obeys the associative and distributive laws of algebra, leading to

$$(r+s)(\mathbf{A}+\mathbf{B}) = r(\mathbf{A}+\mathbf{B}) + s(\mathbf{A}+\mathbf{B}) = r\mathbf{A} + r\mathbf{B} + s\mathbf{A} + s\mathbf{B}$$

Division of a vector by a scalar is merely multiplication by the reciprocal of that scalar. The multiplication of a vector by a vector is discussed in Sections 1.6 and 1.7. Two vectors are said to be equal if their difference is zero, or $\mathbf{A} = \mathbf{B}$ if $\mathbf{A} - \mathbf{B} = 0$.

In our use of vector fields we shall always add and subtract vectors that are defined at the same point. For example, the *total* magnetic field about a small horseshoe magnet will be shown to be the sum of the fields produced by the earth and the permanent magnet; the total field at any point is the sum of the individual fields at that point.

If we are not considering a vector *field*, we may add or subtract vectors that are not defined at the same point. For example, the sum of the gravitational force acting on a 150 lb_f (pound-force) man at the North Pole and that acting on a 175 lb_f person at the South Pole may be obtained by shifting each force vector to the South Pole before addition. The result is a force of 25 lb_f directed toward the center of the earth at the South Pole; if we wanted to be difficult, we could just as well describe the force as 25 lb_f directed *away* from the center of the earth (or "upward") at the North Pole.[2]

1.3 THE RECTANGULAR COORDINATE SYSTEM

To describe a vector accurately, some specific lengths, directions, angles, projections, or components must be given. There are three simple methods of doing this, and about eight or ten other methods that are useful in very special cases. We are going

[2] Students have argued that the force might be described at the equator as being in a "northerly" direction. They are right, but enough is enough.

to use only the three simple methods, and the simplest of these is the *rectangular*, or *rectangular cartesian*, coordinate system.

In the rectangular coordinate system we set up three coordinate axes mutually at right angles to each other and call them the x, y, and z axes. It is customary to choose a *right-handed* coordinate system, in which a rotation (through the smaller angle) of the x axis into the y axis would cause a right-handed screw to progress in the direction of the z axis. If the right hand is used, then the thumb, forefinger, and middle finger may be identified, respectively, as the x, y, and z axes. Figure 1.2*a* shows a right-handed rectangular coordinate system.

A point is located by giving its x, y, and z coordinates. These are, respectively, the distances from the origin to the intersection of perpendicular lines dropped from the point to the x, y, and z axes. An alternative method of interpreting coordinate

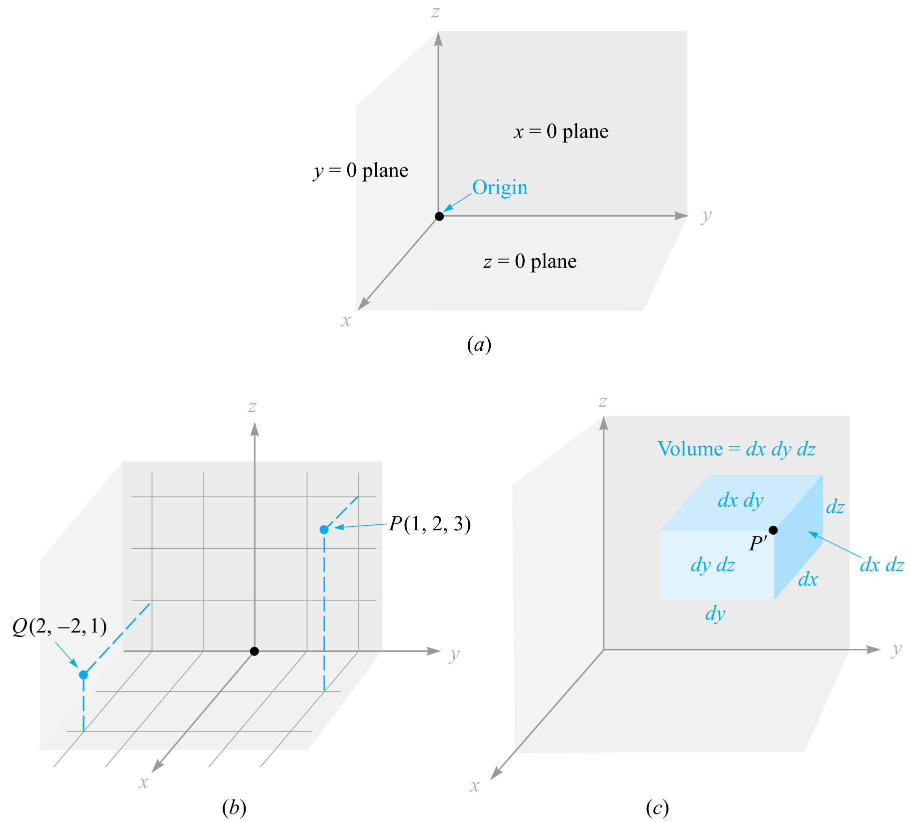

Figure 1.2 (*a*) A right-handed rectangular coordinate system. If the curved fingers of the right hand indicate the direction through which the x axis is turned into coincidence with the y axis, the thumb shows the direction of the z axis. (*b*) The location of points $P(1, 2, 3)$ and $Q(2, -2, 1)$. (*c*) The differential volume element in rectangular coordinates; dx, dy, and dz are, in general, independent differentials.

values, which *must* be used in all other coordinate systems, is to consider the point as being at the common intersection of three surfaces. These are the planes $x =$ constant, $y =$ constant, and $z =$ constant, where the constants are the coordinate values of the point.

Figure 1.2*b* shows points P and Q whose coordinates are (1, 2, 3) and (2, −2, 1), respectively. Point P is therefore located at the common point of intersection of the planes $x = 1$, $y = 2$, and $z = 3$, whereas point Q is located at the intersection of the planes $x = 2$, $y = -2$, and $z = 1$.

As we encounter other coordinate systems in Sections 1.8 and 1.9, we expect points to be located at the common intersection of three surfaces, not necessarily planes, but still mutually perpendicular at the point of intersection.

If we visualize three planes intersecting at the general point P, whose coordinates are x, y, and z, we may increase each coordinate value by a differential amount and obtain three slightly displaced planes intersecting at point P', whose coordinates are $x + dx$, $y + dy$, and $z + dz$. The six planes define a rectangular parallelepiped whose volume is $dv = dxdydz$; the surfaces have differential areas dS of $dxdy$, $dydz$, and $dzdx$. Finally, the distance dL from P to P' is the diagonal of the parallelepiped and has a length of $\sqrt{(dx)^2 + (dy)^2 + (dz)^2}$. The volume element is shown in Figure 1.2*c*; point P' is indicated, but point P is located at the only invisible corner.

All this is familiar from trigonometry or solid geometry and as yet involves only scalar quantities. We will describe vectors in terms of a coordinate system in the next section.

1.4 VECTOR COMPONENTS AND UNIT VECTORS

To describe a vector in the rectangular coordinate system, let us first consider a vector $\mathbf{r}$ extending outward from the origin. A logical way to identify this vector is by giving the three *component vectors*, lying along the three coordinate axes, whose vector sum must be the given vector. If the component vectors of the vector $\mathbf{r}$ are $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, then $\mathbf{r} = \mathbf{x} + \mathbf{y} + \mathbf{z}$. The component vectors are shown in Figure 1.3*a*. Instead of one vector, we now have three, but this is a step forward because the three vectors are of a very simple nature; each is always directed along one of the coordinate axes.

The component vectors have magnitudes that depend on the given vector (such as $\mathbf{r}$), but they each have a known and constant direction. This suggests the use of *unit vectors* having unit magnitude by definition; these are parallel to the coordinate axes and they point in the direction of increasing coordinate values. We reserve the symbol $\mathbf{a}$ for a unit vector and identify its direction by an appropriate subscript. Thus $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are the unit vectors in the rectangular coordinate system.[3] They are directed along the x, y, and z axes, respectively, as shown in Figure 1.3*b*.

If the component vector $\mathbf{y}$ happens to be two units in magnitude and directed toward increasing values of y, we should then write $\mathbf{y} = 2\mathbf{a}_y$. A vector $\mathbf{r}_P$ pointing

[3] The symbols **i**, **j**, and **k** are also commonly used for the unit vectors in rectangular coordinates.

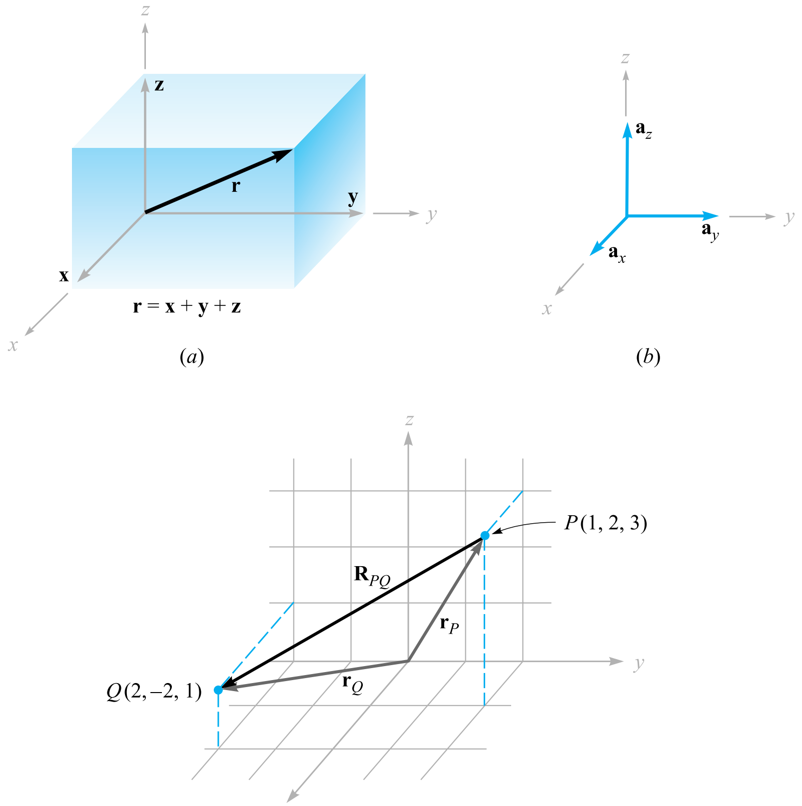

Figure 1.3 (*a*) The component vectors **x**, **y**, and **z** of vector **r**. (*b*) The unit vectors of the rectangular coordinate system have unit magnitude and are directed toward increasing values of their respective variables. (*c*) The vector $\mathbf{R}_{PQ}$ is equal to the vector difference $\mathbf{r}_Q - \mathbf{r}_P$.

from the origin to point $P(1, 2, 3)$ is written $\mathbf{r}_P = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$. The vector from P to Q may be obtained by applying the rule of vector addition. This rule shows that the vector from the origin to P plus the vector from P to Q is equal to the vector from the origin to Q. The desired vector from $P(1, 2, 3)$ to $Q(2, -2, 1)$ is therefore

$$\mathbf{R}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P = (2-1)\mathbf{a}_x + (-2-2)\mathbf{a}_y + (1-3)\mathbf{a}_z$$
$$= \mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z$$

The vectors $\mathbf{r}_P$, $\mathbf{r}_Q$, and $\mathbf{R}_{PQ}$ are shown in Figure 1.3*c*.

The last vector does not extend outward from the origin, as did the vector **r** we initially considered. However, we have already learned that vectors having the same magnitude and pointing in the same direction are equal, so we see that to help our visualization processes we are at liberty to slide any vector over to the origin before determining its component vectors. Parallelism must, of course, be maintained during the sliding process.

If we are discussing a force vector **F**, or indeed any vector other than a displacement-type vector such as **r**, the problem arises of providing suitable letters for the three component vectors. It would not do to call them **x**, **y**, and **z**, for these are displacements, or directed distances, and are measured in meters (abbreviated m) or some other unit of length. The problem is most often avoided by using *component scalars*, simply called *components*, F_x, F_y, and F_z. The components are the signed magnitudes of the component vectors. We may then write $\mathbf{F} = F_x\mathbf{a}_x + F_y\mathbf{a}_y + F_z\mathbf{a}_z$. The component vectors are $F_x\mathbf{a}_x$, $F_y\mathbf{a}_y$, and $F_z\mathbf{a}_z$.

Any vector **B** then may be described by $\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$. The magnitude of **B** written $|\mathbf{B}|$ or simply B, is given by

$$|\mathbf{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2} \tag{1}$$

Each of the three coordinate systems we discuss will have its three fundamental and mutually perpendicular unit vectors that are used to resolve any vector into its component vectors. Unit vectors are not limited to this application. It is helpful to write a unit vector having a specified direction. This is easily done, for a unit vector in a given direction is merely a vector in that direction divided by its magnitude. A unit vector in the **r** direction is $\mathbf{r}/\sqrt{x^2 + y^2 + z^2}$, and a unit vector in the direction of the vector **B** is

$$\mathbf{a}_B = \frac{\mathbf{B}}{\sqrt{B_x^2 + B_y^2 + B_z^2}} = \frac{\mathbf{B}}{|\mathbf{B}|} \tag{2}$$

EXAMPLE 1.1

Specify the unit vector extending from the origin toward the point $G(2, -2, -1)$.

Solution. We first construct the vector extending from the origin to point G,

$$\mathbf{G} = 2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z$$

We continue by finding the magnitude of **G**,

$$|\mathbf{G}| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = 3$$

and finally expressing the desired unit vector as the quotient,

$$\mathbf{a}_G = \frac{\mathbf{G}}{|\mathbf{G}|} = \tfrac{2}{3}\mathbf{a}_x - \tfrac{2}{3}\mathbf{a}_y - \tfrac{1}{3}\mathbf{a}_z = 0.667\mathbf{a}_x - 0.667\mathbf{a}_y - 0.333\mathbf{a}_z$$

A special symbol is desirable for a unit vector so that its character is immediately apparent. Symbols that have been used are $\mathbf{u}_B$, $\mathbf{a}_B$, $\mathbf{1_B}$, or even **b**. We will consistently use the lowercase **a** with an appropriate subscript.

[NOTE: Throughout the text, drill problems appear following sections in which a new principle is introduced in order to allow students to test their understanding of the basic fact itself. The problems are useful in gaining familiarity with new terms and ideas and should all be worked. More general problems appear at the ends of the chapters. The answers to the drill problems are given in the same order as the parts of the problem.]

D1.1. Given points $M(-1, 2, 1)$, $N(3, -3, 0)$, and $P(-2, -3, -4)$, find: (*a*) $\mathbf{R}_{MN}$; (*b*) $\mathbf{R}_{MN} + \mathbf{R}_{MP}$; (*c*) $|\mathbf{r}_M|$; (*d*) $\mathbf{a}_{MP}$; (*e*) $|2\mathbf{r}_P - 3\mathbf{r}_N|$.

Ans. $4\mathbf{a}_x - 5\mathbf{a}_y - \mathbf{a}_z$; $3\mathbf{a}_x - 10\mathbf{a}_y - 6\mathbf{a}_z$; 2.45; $-0.14\mathbf{a}_x - 0.7\mathbf{a}_y - 0.7\mathbf{a}_z$; 15.56

1.5 THE VECTOR FIELD

We have defined a vector field as a vector function of a position vector. In general, the magnitude and direction of the function will change as we move throughout the region, and the value of the vector function must be determined using the coordinate values of the point in question. Because we have considered only the rectangular coordinate system, we expect the vector to be a function of the variables x, y, and z.

If we again represent the position vector as $\mathbf{r}$, then a vector field $\mathbf{G}$ can be expressed in functional notation as $\mathbf{G}(\mathbf{r})$; a scalar field T is written as $T(\mathbf{r})$.

If we inspect the velocity of the water in the ocean in some region near the surface where tides and currents are important, we might decide to represent it by a velocity vector that is in any direction, even up or down. If the z axis is taken as upward, the x axis in a northerly direction, the y axis to the west, and the origin at the surface, we have a right-handed coordinate system and may write the velocity vector as $\mathbf{v} = v_x\mathbf{a}_x + v_y\mathbf{a}_y + v_z\mathbf{a}_z$, or $\mathbf{v}(\mathbf{r}) = v_x(\mathbf{r})\mathbf{a}_x + v_y(\mathbf{r})\mathbf{a}_y + v_z(\mathbf{r})\mathbf{a}_z$; each of the components v_x, v_y, and v_z may be a function of the three variables x, y, and z. If we are in some portion of the Gulf Stream where the water is moving only to the north, then v_y and v_z are zero. Further simplifying assumptions might be made if the velocity falls off with depth and changes very slowly as we move north, south, east, or west. A suitable expression could be $\mathbf{v} = 2e^{z/100}\mathbf{a}_x$. We have a velocity of 2 m/s (meters per second) at the surface and a velocity of 0.368×2, or 0.736 m/s, at a depth of 100 m ($z = -100$). The velocity continues to decrease with depth, while maintaining a constant direction.

D1.2. A vector field $\mathbf{S}$ is expressed in rectangular coordinates as $\mathbf{S} = \{125/[(x-1)^2+(y-2)^2+(z+1)^2]\}\{(x-1)\mathbf{a}_x+(y-2)\mathbf{a}_y+(z+1)\mathbf{a}_z\}$. (*a*) Evaluate $\mathbf{S}$ at $P(2, 4, 3)$. (*b*) Determine a unit vector that gives the direction of $\mathbf{S}$ at P. (*c*) Specify the surface $f(x, y, z)$ on which $|\mathbf{S}| = 1$.

Ans. $5.95\mathbf{a}_x + 11.90\mathbf{a}_y + 23.8\mathbf{a}_z$; $0.218\mathbf{a}_x + 0.436\mathbf{a}_y + 0.873\mathbf{a}_z$; $\sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2} = 125$

1.6 THE DOT PRODUCT

We now consider the first of two types of vector multiplication. The second type will be discussed in the following section.

Given two vectors **A** and **B**, the *dot product*, or *scalar product*, is defined as the product of the magnitude of **A**, the magnitude of **B**, and the cosine of the smaller angle between them,

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta_{AB} \tag{3}$$

The dot appears between the two vectors and should be made heavy for emphasis. The dot, or scalar, product is a scalar, as one of the names implies, and it obeys the commutative law,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{4}$$

for the sign of the angle does not affect the cosine term. The expression $\mathbf{A} \cdot \mathbf{B}$ is read "**A** dot **B**."

Perhaps the most common application of the dot product is in mechanics, where a constant force **F** applied over a straight displacement **L** does an amount of work $FL\cos\theta$, which is more easily written $\mathbf{F} \cdot \mathbf{L}$. We might anticipate one of the results of Chapter 4 by pointing out that if the force varies along the path, integration is necessary to find the total work, and the result becomes

$$\text{Work} = \int \mathbf{F} \cdot d\mathbf{L}$$

Another example might be taken from magnetic fields. The total flux Φ crossing a surface of area S is given by BS if the magnetic flux density B is perpendicular to the surface and uniform over it. We define a *vector surface* **S** as having area for its magnitude and having a direction *normal* to the surface (avoiding for the moment the problem of which of the two possible normals to take). The flux crossing the surface is then $\mathbf{B} \cdot \mathbf{S}$. This expression is valid for any direction of the uniform magnetic flux density. If the flux density is not constant over the surface, the total flux is $\Phi = \int \mathbf{B} \cdot d\mathbf{S}$. Integrals of this general form appear in Chapter 3 when we study electric flux density.

Finding the angle between two vectors in three-dimensional space is often a job we would prefer to avoid, and for that reason the definition of the dot product is usually not used in its basic form. A more helpful result is obtained by considering two vectors whose rectangular components are given, such as $\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$ and $\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$. The dot product also obeys the distributive law, and, therefore, $\mathbf{A} \cdot \mathbf{B}$ yields the sum of nine scalar terms, each involving the dot product of two unit vectors. Because the angle between two different unit vectors of the rectangular coordinate system is 90°, we then have

$$\mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_y \cdot \mathbf{a}_x = \mathbf{a}_x \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_x = \mathbf{a}_y \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_y = 0$$

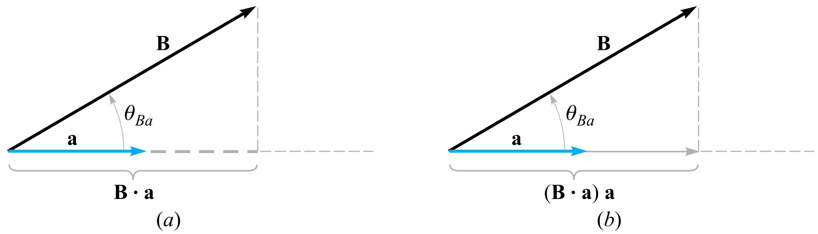

Figure 1.4 (*a*) The scalar component of B in the direction of the unit vector a is B · a. (*b*) The vector component of B in the direction of the unit vector a is (B · a)a.

The remaining three terms involve the dot product of a unit vector with itself, which is unity, giving finally

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \tag{5}$$

which is an expression involving no angles.

A vector dotted with itself yields the magnitude squared, or

$$\mathbf{A} \cdot \mathbf{A} = A^2 = |\mathbf{A}|^2 \tag{6}$$

and any unit vector dotted with itself is unity,

$$\mathbf{a}_A \cdot \mathbf{a}_A = 1$$

One of the most important applications of the dot product is that of finding the component of a vector in a given direction. Referring to Figure 1.4*a*, we can obtain the component (scalar) of **B** in the direction specified by the unit vector **a** as

$$\mathbf{B} \cdot \mathbf{a} = |\mathbf{B}|\,|\mathbf{a}| \cos \theta_{Ba} = |\mathbf{B}| \cos \theta_{Ba}$$

The sign of the component is positive if $0 \le \theta_{Ba} \le 90^\circ$ and negative whenever $90^\circ \le \theta_{Ba} \le 180^\circ$.

To obtain the component *vector* of **B** in the direction of **a**, we multiply the component (scalar) by **a**, as illustrated by Figure 1.4*b*. For example, the component of **B** in the direction of $\mathbf{a}_x$ is $\mathbf{B} \cdot \mathbf{a}_x = B_x$, and the component vector is $B_x \mathbf{a}_x$, or $(\mathbf{B} \cdot \mathbf{a}_x)\mathbf{a}_x$. Hence, the problem of finding the component of a vector in any direction becomes the problem of finding a unit vector in that direction, and that we can do.

The geometrical term *projection* is also used with the dot product. Thus, $\mathbf{B} \cdot \mathbf{a}$ is the projection of **B** in the **a** direction.

EXAMPLE 1.2

In order to illustrate these definitions and operations, consider the vector field $\mathbf{G} = y\mathbf{a}_x - 2.5x\mathbf{a}_y + 3\mathbf{a}_z$ and the point $Q(4, 5, 2)$. We wish to find: **G** at Q; the scalar component of **G** at Q in the direction of $\mathbf{a}_N = \frac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z)$; the vector component of **G** at Q in the direction of $\mathbf{a}_N$; and finally, the angle θ_{Ga} between $\mathbf{G}(\mathbf{r}_Q)$ and $\mathbf{a}_N$.

Solution. Substituting the coordinates of point Q into the expression for $\mathbf{G}$, we have

$$\mathbf{G}(\mathbf{r}_Q) = 5\mathbf{a}_x - 10\mathbf{a}_y + 3\mathbf{a}_z$$

Next we find the scalar component. Using the dot product, we have

$$\mathbf{G}\cdot\mathbf{a}_N = (5\mathbf{a}_x - 10\mathbf{a}_y + 3\mathbf{a}_z)\cdot\tfrac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z) = \tfrac{1}{3}(10 - 10 - 6) = -2$$

The vector component is obtained by multiplying the scalar component by the unit vector in the direction of $\mathbf{a}_N$,

$$(\mathbf{G}\cdot\mathbf{a}_N)\mathbf{a}_N = -(2)\tfrac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z) = -1.333\mathbf{a}_x - 0.667\mathbf{a}_y + 1.333\mathbf{a}_z$$

The angle between $\mathbf{G}(\mathbf{r}_Q)$ and $\mathbf{a}_N$ is found from

$$\mathbf{G}\cdot\mathbf{a}_N = |\mathbf{G}|\cos\theta_{Ga}$$
$$-2 = \sqrt{25+100+9}\cos\theta_{Ga}$$

and

$$\theta_{Ga} = \cos^{-1}\frac{-2}{\sqrt{134}} = 99.9^\circ$$

D1.3. The three vertices of a triangle are located at $A(6, -1, 2)$, $B(-2, 3, -4)$, and $C(-3, 1, 5)$. Find: (*a*) $\mathbf{R}_{AB}$; (*b*) $\mathbf{R}_{AC}$; (*c*) the angle θ_{BAC} at vertex A; (*d*) the (vector) projection of $\mathbf{R}_{AB}$ on $\mathbf{R}_{AC}$.

Ans. $-8\mathbf{a}_x + 4\mathbf{a}_y - 6\mathbf{a}_z$; $-9\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$; 53.6°; $-5.94\mathbf{a}_x + 1.319\mathbf{a}_y + 1.979\mathbf{a}_z$

1.7 THE CROSS PRODUCT

Given two vectors $\mathbf{A}$ and $\mathbf{B}$, we now define the *cross product*, or *vector product*, of $\mathbf{A}$ and $\mathbf{B}$, written with a cross between the two vectors as $\mathbf{A}\times\mathbf{B}$ and read "**A** cross **B**." The cross product $\mathbf{A}\times\mathbf{B}$ is a vector; the magnitude of $\mathbf{A}\times\mathbf{B}$ is equal to the product of the magnitudes of **A**, **B**, and the sine of the smaller angle between **A** and **B**; the direction of $\mathbf{A}\times\mathbf{B}$ is perpendicular to the plane containing **A** and **B** and is along one of the two possible perpendiculars which is in the direction of advance of a right-handed screw as **A** is turned into **B**. This direction is illustrated in Figure 1.5. Remember that either vector may be moved about at will, maintaining its direction constant, until the two vectors have a "common origin." This determines the plane containing both. However, in most of our applications we will be concerned with vectors defined at the same point.

As an equation we can write

$$\boxed{\mathbf{A}\times\mathbf{B} = \mathbf{a}_N|\mathbf{A}|\,|\mathbf{B}|\sin\theta_{AB}} \qquad (7)$$

where an additional statement, such as that given above, is required to explain the direction of the unit vector $\mathbf{a}_N$. The subscript stands for "normal."

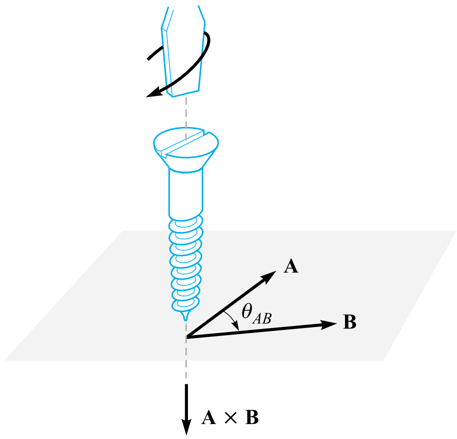

Figure 1.5 The direction of A × B is in the direction of advance of a right-handed screw as A is turned into B.

Reversing the order of the vectors **A** and **B** results in a unit vector in the opposite direction, and we see that the cross product is not commutative, for $\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B})$. If the definition of the cross product is applied to the unit vectors $\mathbf{a}_x$ and $\mathbf{a}_y$, we find $\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$, for each vector has unit magnitude, the two vectors are perpendicular, and the rotation of $\mathbf{a}_x$ into $\mathbf{a}_y$ indicates the positive z direction by the definition of a right-handed coordinate system. In a similar way, $\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x$ and $\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$. Note the alphabetic symmetry. As long as the three vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are written in order (and assuming that $\mathbf{a}_x$ follows $\mathbf{a}_z$, like three elephants in a circle holding tails, so that we could also write $\mathbf{a}_y$, $\mathbf{a}_z$, $\mathbf{a}_x$ or $\mathbf{a}_z$, $\mathbf{a}_x$, $\mathbf{a}_y$), then the cross and equal sign may be placed in either of the two vacant spaces. As a matter of fact, it is now simpler to define a right-handed rectangular coordinate system by saying that $\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$.

A simple example of the use of the cross product may be taken from geometry or trigonometry. To find the area of a parallelogram, the product of the lengths of two adjacent sides is multiplied by the sine of the angle between them. Using vector notation for the two sides, we then may express the (scalar) area as the *magnitude* of $\mathbf{A} \times \mathbf{B}$, or $|\mathbf{A} \times \mathbf{B}|$.

The cross product may be used to replace the right-hand rule familiar to all electrical engineers. Consider the force on a straight conductor of length **L**, where the direction assigned to **L** corresponds to the direction of the steady current I, and a uniform magnetic field of flux density **B** is present. Using vector notation, we may write the result neatly as $\mathbf{F} = I\mathbf{L} \times \mathbf{B}$. This relationship will be obtained later in Chapter 9.

The evaluation of a cross product by means of its definition turns out to be more work than the evaluation of the dot product from its definition, for not only must we find the angle between the vectors, but we must also find an expression for the unit vector $\mathbf{a}_N$. This work may be avoided by using rectangular components for the two vectors **A** and **B** and expanding the cross product as a sum of nine simpler cross products, each involving two unit vectors,

$$\begin{aligned}\mathbf{A} \times \mathbf{B} = & A_x B_x \mathbf{a}_x \times \mathbf{a}_x + A_x B_y \mathbf{a}_x \times \mathbf{a}_y + A_x B_z \mathbf{a}_x \times \mathbf{a}_z \\ & + A_y B_x \mathbf{a}_y \times \mathbf{a}_x + A_y B_y \mathbf{a}_y \times \mathbf{a}_y + A_y B_z \mathbf{a}_y \times \mathbf{a}_z \\ & + A_z B_x \mathbf{a}_z \times \mathbf{a}_x + A_z B_y \mathbf{a}_z \times \mathbf{a}_y + A_z B_z \mathbf{a}_z \times \mathbf{a}_z\end{aligned}$$

We have already found that $\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$, $\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x$, and $\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$. The three remaining terms are zero, for the cross product of any vector with itself is zero, since the included angle is zero. These results may be combined to give

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{a}_x + (A_z B_x - A_x B_z)\mathbf{a}_y + (A_x B_y - A_y B_x)\mathbf{a}_z \tag{8}$$

or written as a determinant in a more easily remembered form,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{9}$$

Thus, if $\mathbf{A} = 2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$ and $\mathbf{B} = -4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z$, we have

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -3 & 1 \\ -4 & -2 & 5 \end{vmatrix} \\ &= [(-3)(5) - (1(-2)]\mathbf{a}_x - [(2)(5) - (1)(-4)]\mathbf{a}_y + [(2)(-2) - (-3)(-4)]\mathbf{a}_z \\ &= -13\mathbf{a}_x - 14\mathbf{a}_y - 16\mathbf{a}_z\end{aligned}$$

D1.4. The three vertices of a triangle are located at $A(6, -1, 2)$, $B(-2, 3, -4)$, and $C(-3, 1, 5)$. Find: (*a*) $\mathbf{R}_{AB} \times \mathbf{R}_{AC}$; (*b*) the area of the triangle; (*c*) a unit vector perpendicular to the plane in which the triangle is located.

Ans. $24\mathbf{a}_x + 78\mathbf{a}_y + 20\mathbf{a}_z$; 42.0; $0.286\mathbf{a}_x + 0.928\mathbf{a}_y + 0.238\mathbf{a}_z$

1.8 OTHER COORDINATE SYSTEMS: CIRCULAR CYLINDRICAL COORDINATES

The rectangular coordinate system is generally the one in which students prefer to work every problem. This often means a lot more work, because many problems possess a type of symmetry that pleads for a more logical treatment. It is easier to do now, once and for all, the work required to become familiar with cylindrical and spherical coordinates, instead of applying an equal or greater effort to every problem involving cylindrical or spherical symmetry later. With this in mind, we will take a careful and unhurried look at cylindrical and spherical coordinates.

The circular cylindrical coordinate system is the three-dimensional version of the polar coordinates of analytic geometry. In polar coordinates, a point is located in a plane by giving both its distance ρ from the origin and the angle ϕ between the line from the point to the origin and an arbitrary radial line, taken as $\phi = 0$.[4] In circular cylindrical coordinates, we also specify the distance z of the point from an arbitrary $z = 0$ reference plane that is perpendicular to the line $\rho = 0$. For simplicity, we usually refer to circular cylindrical coordinates simply as cylindrical coordinates. This will not cause any confusion in reading this book, but it is only fair to point out that there are such systems as elliptic cylindrical coordinates, hyperbolic cylindrical coordinates, parabolic cylindrical coordinates, and others.

We no longer set up three axes as with rectangular coordinates, but we must instead consider any point as the intersection of three mutually perpendicular surfaces. These surfaces are a circular cylinder ($\rho =$ constant), a plane ($\phi =$ constant), and another plane ($z =$ constant). This corresponds to the location of a point in a rectangular coordinate system by the intersection of three planes ($x =$ constant, $y =$ constant, and $z =$ constant). The three surfaces of circular cylindrical coordinates are shown in Figure 1.6*a*. Note that three such surfaces may be passed through any point, unless it lies on the z axis, in which case one plane suffices.

Three unit vectors must also be defined, but we may no longer direct them along the "coordinate axes," for such axes exist only in rectangular coordinates. Instead, we take a broader view of the unit vectors in rectangular coordinates and realize that they are directed toward increasing coordinate values and are perpendicular to the surface on which that coordinate value is constant (i.e., the unit vector $\mathbf{a}_x$ is normal to the plane $x =$ constant and points toward larger values of x). In a corresponding way we may now define three unit vectors in cylindrical coordinates, $\mathbf{a}_\rho$, $\mathbf{a}_\phi$, and $\mathbf{a}_z$.

The unit vector $\mathbf{a}_\rho$ at a point $P(\rho_1, \phi_1, z_1)$ is directed radially outward, normal to the cylindrical surface $\rho = \rho_1$. It lies in the planes $\phi = \phi_1$ and $z = z_1$. The unit vector $\mathbf{a}_\phi$ is normal to the plane $\phi = \phi_1$, points in the direction of increasing ϕ, lies in the plane $z = z_1$, and is tangent to the cylindrical surface $\rho = \rho_1$. The unit vector $\mathbf{a}_z$ is the same as the unit vector $\mathbf{a}_z$ of the rectangular coordinate system. Figure 1.6*b* shows the three vectors in cylindrical coordinates.

In rectangular coordinates, the unit vectors are not functions of the coordinates. Two of the unit vectors in cylindrical coordinates, $\mathbf{a}_\rho$ and $\mathbf{a}_\phi$, however, *do* vary with the coordinate ϕ, as their directions change. In integration or differentiation with respect to ϕ, then, $\mathbf{a}_\rho$ and $\mathbf{a}_\phi$ must not be treated as constants.

The unit vectors are again mutually perpendicular, for each is normal to one of the three mutually perpendicular surfaces, and we may define a right-handed cylindrical

[4] The two variables of polar coordinates are commonly called r and θ. With three coordinates, however, it is more common to use ρ for the radius variable of cylindrical coordinates and r for the (different) radius variable of spherical coordinates. Also, the angle variable of cylindrical coordinates is customarily called ϕ because everyone uses θ for a different angle in spherical coordinates. The angle ϕ is common to both cylindrical and spherical coordinates. See?

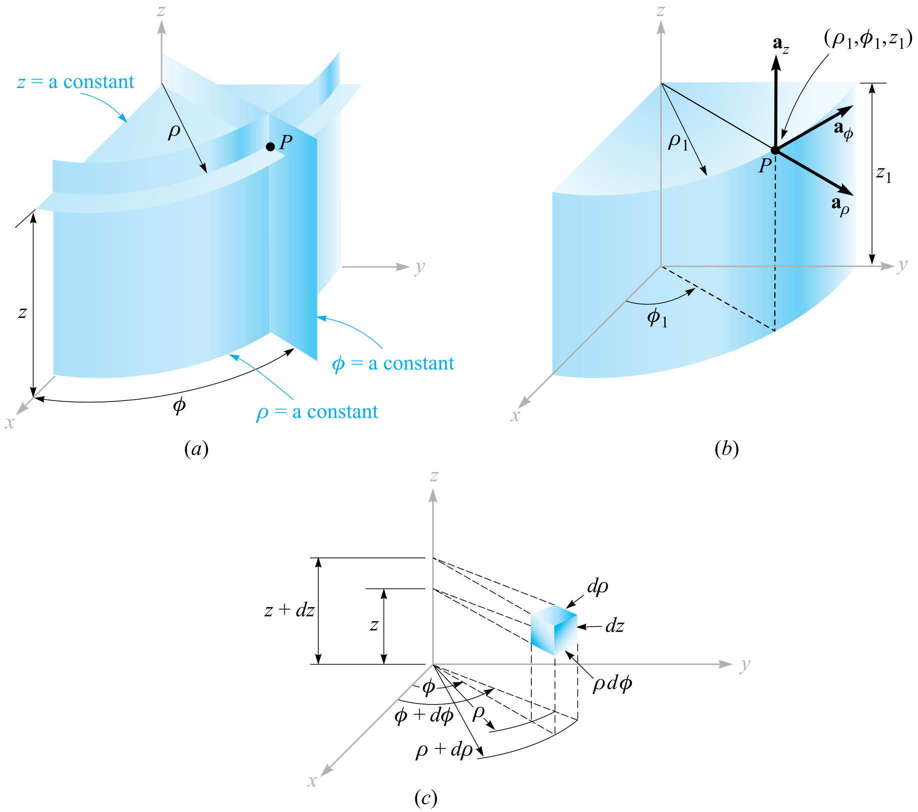

Figure 1.6 (*a*) The three mutually perpendicular surfaces of the circular cylindrical coordinate system. (*b*) The three unit vectors of the circular cylindrical coordinate system. (*c*) The differential volume unit in the circular cylindrical coordinate system; $d\rho$, $\rho d\phi$, and dz are all elements of length.

coordinate system as one in which $\mathbf{a}_\rho \times \mathbf{a}_\phi = \mathbf{a}_z$, or (for those who have flexible fingers) as one in which the thumb, forefinger, and middle finger point in the direction of increasing ρ, ϕ, and z, respectively.

A differential volume element in cylindrical coordinates may be obtained by increasing ρ, ϕ, and z by the differential increments $d\rho$, $d\phi$, and dz. The two cylinders of radius ρ and $\rho + d\rho$, the two radial planes at angles ϕ and $\phi + d\phi$, and the two "horizontal" planes at "elevations" z and $z + dz$ now enclose a small volume, as shown in Figure 1.6*c*, having the shape of a truncated wedge. As the volume element becomes very small, its shape approaches that of a rectangular parallelepiped having sides of length $d\rho$, $\rho d\phi$, and dz. Note that $d\rho$ and dz are dimensionally lengths, but $d\phi$ is not; $\rho d\phi$ is the length. The surfaces have areas of $\rho\, d\rho\, d\phi$, $d\rho\, dz$, and $\rho\, d\phi\, dz$, and the volume becomes $\rho\, d\rho\, d\phi\, dz$.

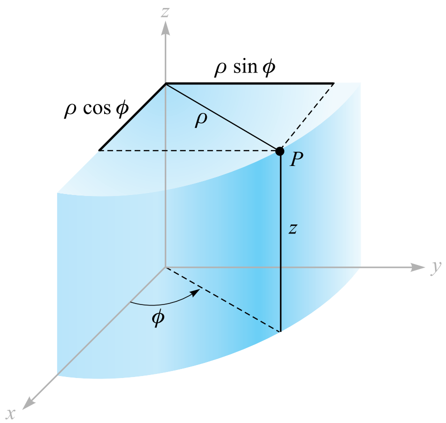

Figure 1.7 The relationship between the rectangular variables x, y, z and the cylindrical coordinate variables ρ, ϕ, z. There is no change in the variable z between the two systems.

The variables of the rectangular and cylindrical coordinate systems are easily related to each other. Referring to Figure 1.7, we see that

$$\begin{aligned} x &= \rho \cos \phi \\ y &= \rho \sin \phi \\ z &= z \end{aligned} \tag{10}$$

From the other viewpoint, we may express the cylindrical variables in terms of x, y, and z:

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2} \quad (\rho \geq 0) \\ \phi &= \tan^{-1} \frac{y}{x} \\ z &= z \end{aligned} \tag{11}$$

We consider the variable ρ to be positive or zero, thus using only the positive sign for the radical in (11). The proper value of the angle ϕ is determined by inspecting the signs of x and y. Thus, if $x = -3$ and $y = 4$, we find that the point lies in the second quadrant so that $\rho = 5$ and $\phi = 126.9^\circ$. For $x = 3$ and $y = -4$, we have $\phi = -53.1^\circ$ or 306.9°, whichever is more convenient.

Using (10) or (11), scalar functions given in one coordinate system are easily transformed into the other system.

A vector function in one coordinate system, however, requires two steps in order to transform it to another coordinate system, because a different set of component vectors is generally required. That is, we may be given a rectangular vector

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

where each component is given as a function of x, y, and z, and we need a vector in cylindrical coordinates

$$\mathbf{A} = A_\rho\mathbf{a}_\rho + A_\phi\mathbf{a}_\phi + A_z\mathbf{a}_z$$

where each component is given as a function of ρ, ϕ, and z.

To find any desired component of a vector, we recall from the discussion of the dot product that a component in a desired direction may be obtained by taking the dot product of the vector and a unit vector in the desired direction. Hence,

$$A_\rho = \mathbf{A}\cdot\mathbf{a}_\rho \qquad \text{and} \qquad A_\phi = \mathbf{A}\cdot\mathbf{a}_\phi$$

Expanding these dot products, we have

$$A_\rho = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)\cdot\mathbf{a}_\rho = A_x\mathbf{a}_x\cdot\mathbf{a}_\rho + A_y\mathbf{a}_y\cdot\mathbf{a}_\rho \tag{12}$$

$$A_\phi = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)\cdot\mathbf{a}_\phi = A_x\mathbf{a}_x\cdot\mathbf{a}_\phi + A_y\mathbf{a}_y\cdot\mathbf{a}_\phi \tag{13}$$

and

$$A_z = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)\cdot\mathbf{a}_z = A_z\mathbf{a}_z\cdot\mathbf{a}_z = A_z \tag{14}$$

since $\mathbf{a}_z\cdot\mathbf{a}_\rho$ and $\mathbf{a}_z\cdot\mathbf{a}_\phi$ are zero.

In order to complete the transformation of the components, it is necessary to know the dot products $\mathbf{a}_x\cdot\mathbf{a}_\rho$, $\mathbf{a}_y\cdot\mathbf{a}_\rho$, $\mathbf{a}_x\cdot\mathbf{a}_\phi$, and $\mathbf{a}_y\cdot\mathbf{a}_\phi$. Applying the definition of the dot product, we see that since we are concerned with unit vectors, the result is merely the cosine of the angle between the two unit vectors in question. Referring to Figure 1.7 and thinking mightily, we identify the angle between $\mathbf{a}_x$ and $\mathbf{a}_\rho$ as ϕ, and thus $\mathbf{a}_x\cdot\mathbf{a}_\rho = \cos\phi$, but the angle between $\mathbf{a}_y$ and $\mathbf{a}_\rho$ is $90^\circ - \phi$, and $\mathbf{a}_y\cdot\mathbf{a}_\rho = \cos(90^\circ - \phi) = \sin\phi$. The remaining dot products of the unit vectors are found in a similar manner, and the results are tabulated as functions of ϕ in Table 1.1.

Transforming vectors from rectangular to cylindrical coordinates or vice versa is therefore accomplished by using (10) or (11) to change variables, and by using the dot products of the unit vectors given in Table 1.1 to change components. The two steps may be taken in either order.

Table 1.1 Dot products of unit vectors in cylindrical and rectangular coordinate systems

	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x\cdot$	$\cos\phi$	$-\sin\phi$	0
$\mathbf{a}_y\cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z\cdot$	0	0	1

EXAMPLE 1.3

Transform the vector $\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$ into cylindrical coordinates.

Solution. The new components are

$$
\begin{aligned}
B_\rho &= \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho) \\
&= y \cos\phi - x \sin\phi = \rho \sin\phi \cos\phi - \rho \cos\phi \sin\phi = 0 \\
B_\phi &= \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi) \\
&= -y \sin\phi - x \cos\phi = -\rho \sin^2\phi - \rho \cos^2\phi = -\rho
\end{aligned}
$$

Thus,

$$\mathbf{B} = -\rho\mathbf{a}_\phi + z\mathbf{a}_z$$

D1.5. (*a*) Give the rectangular coordinates of the point $C(\rho = 4.4, \phi = -115^\circ, z = 2)$. (*b*) Give the cylindrical coordinates of the point $D(x = -3.1, y = 2.6, z = -3)$. (*c*) Specify the distance from C to D.

Ans. $C(x = -1.860, y = -3.99, z = 2)$; $D(\rho = 4.05, \phi = 140.0^\circ, z = -3)$; 8.36

D1.6. Transform to cylindrical coordinates: (*a*) $\mathbf{F} = 10\mathbf{a}_x - 8\mathbf{a}_y + 6\mathbf{a}_z$ at point $P(10, -8, 6)$; (*b*) $\mathbf{G} = (2x + y)\mathbf{a}_x - (y - 4x)\mathbf{a}_y$ at point $Q(\rho, \phi, z)$. (*c*) Give the rectangular components of the vector $\mathbf{H} = 20\mathbf{a}_\rho - 10\mathbf{a}_\phi + 3\mathbf{a}_z$ at $P(x = 5, y = 2, z = -1)$.

Ans. $12.81\mathbf{a}_\rho + 6\mathbf{a}_z$; $(2\rho\cos^2\phi - \rho\sin^2\phi + 5\rho\sin\phi\cos\phi)\mathbf{a}_\rho + (4\rho\cos^2\phi - \rho\sin^2\phi - 3\rho\sin\phi\cos\phi)\mathbf{a}_\phi$; $H_x = 22.3$, $H_y = -1.857$, $H_z = 3$

1.9 THE SPHERICAL COORDINATE SYSTEM

We have no two-dimensional coordinate system to help us understand the three-dimensional spherical coordinate system, as we have for the circular cylindrical coordinate system. In certain respects we can draw on our knowledge of the latitude-and-longitude system of locating a place on the surface of the earth, but usually we consider only points on the surface and not those below or above ground.

Let us start by building a spherical coordinate system on the three rectangular axes (Figure 1.8*a*). We first define the distance from the origin to any point as r. The surface $r =$ constant is a sphere.

The second coordinate is an angle θ between the z axis and the line drawn from the origin to the point in question. The surface $\theta =$ constant is a cone, and the two surfaces, cone and sphere, are everywhere perpendicular along their intersection, which is a circle of radius $r \sin\theta$. The coordinate θ corresponds to latitude,

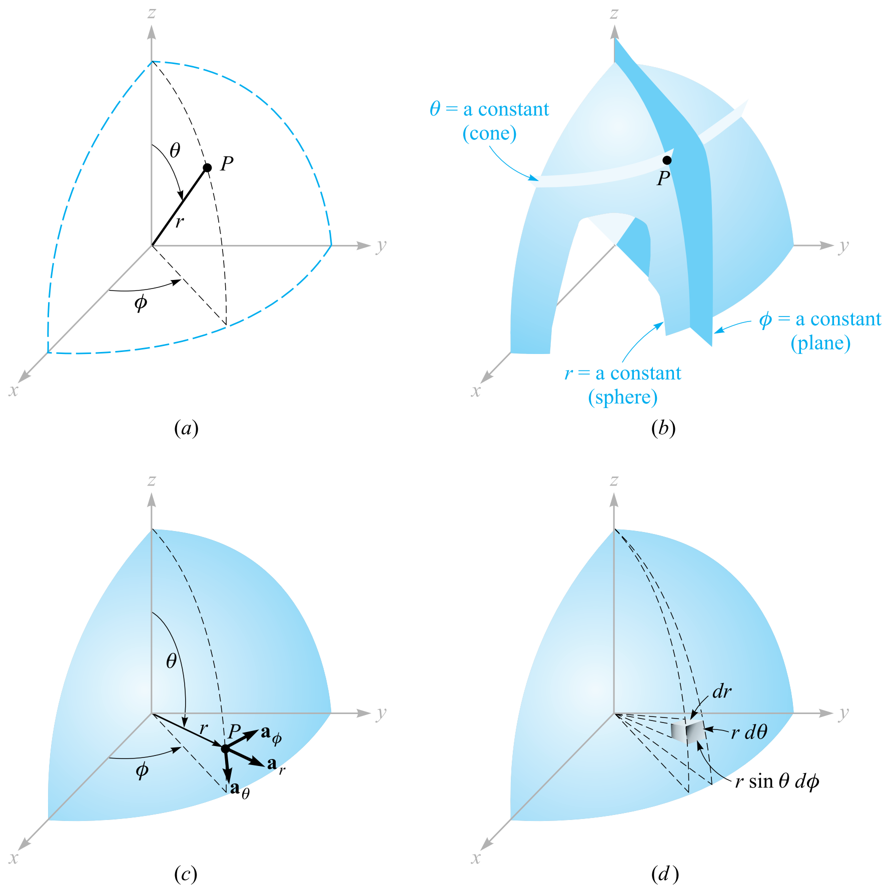

Figure 1.8 (*a*) The three spherical coordinates. (*b*) The three mutually perpendicular surfaces of the spherical coordinate system. (*c*) The three unit vectors of spherical coordinates: $\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$. (*d*) The differential volume element in the spherical coordinate system.

except that latitude is measured from the equator and θ is measured from the "North Pole."

The third coordinate ϕ is also an angle and is exactly the same as the angle ϕ of cylindrical coordinates. It is the angle between the x axis and the projection in the $z = 0$ plane of the line drawn from the origin to the point. It corresponds to the angle of longitude, but the angle ϕ increases to the "east." The surface $\phi =$ constant is a plane passing through the $\theta = 0$ line (or the z axis).

We again consider any point as the intersection of three mutually perpendicular surfaces—a sphere, a cone, and a plane—each oriented in the manner just described. The three surfaces are shown in Figure 1.8*b*.

Three unit vectors may again be defined at any point. Each unit vector is perpendicular to one of the three mutually perpendicular surfaces and oriented in that direction in which the coordinate increases. The unit vector $\mathbf{a}_r$ is directed radially outward, normal to the sphere $r =$ constant, and lies in the cone $\theta =$ constant and the plane $\phi =$ constant. The unit vector $\mathbf{a}_\theta$ is normal to the conical surface, lies in the plane, and is tangent to the sphere. It is directed along a line of "longitude" and points "south." The third unit vector $\mathbf{a}_\phi$ is the same as in cylindrical coordinates, being normal to the plane and tangent to both the cone and the sphere. It is directed to the "east."

The three unit vectors are shown in Figure 1.8c. They are, of course, mutually perpendicular, and a right-handed coordinate system is defined by causing $\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$. Our system is right-handed, as an inspection of Figure 1.8c will show, on application of the definition of the cross product. The right-hand rule identifies the thumb, forefinger, and middle finger with the direction of increasing r, θ, and ϕ, respectively. (Note that the identification in cylindrical coordinates was with ρ, ϕ, and z, and in rectangular coordinates with x, y, and z.) A differential volume element may be constructed in spherical coordinates by increasing r, θ, and ϕ by dr, $d\theta$, and $d\phi$, as shown in Figure 1.8d. The distance between the two spherical surfaces of radius r and $r + dr$ is dr; the distance between the two cones having generating angles of θ and $\theta + d\theta$ is $r d\theta$; and the distance between the two radial planes at angles ϕ and $\phi + d\phi$ is found to be $r \sin\theta d\phi$, after a few moments of trigonometric thought. The surfaces have areas of $r\, dr\, d\theta$, $r \sin\theta\, dr\, d\phi$, and $r^2 \sin\theta\, d\theta\, d\phi$, and the volume is $r^2 \sin\theta\, dr\, d\theta\, d\phi$.

The transformation of scalars from the rectangular to the spherical coordinate system is easily made by using Figure 1.8a to relate the two sets of variables:

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned} \tag{15}$$

The transformation in the reverse direction is achieved with the help of

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} && (r \geq 0) \\ \theta &= \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} && (0^\circ \leq \theta \leq 180^\circ) \\ \phi &= \tan^{-1} \frac{y}{x} \end{aligned} \tag{16}$$

The radius variable r is nonnegative, and θ is restricted to the range from 0° to 180°, inclusive. The angles are placed in the proper quadrants by inspecting the signs of x, y, and z.

The transformation of vectors requires us to determine the products of the unit vectors in rectangular and spherical coordinates. We work out these products from Figure 1.8c and a pinch of trigonometry. Because the dot product of any spherical unit vector with any rectangular unit vector is the component of the spherical

Table 1.2 Dot products of unit vectors in spherical and rectangular coordinate systems

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x \cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y \cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0

vector in the direction of the rectangular vector, the dot products with $\mathbf{a}_z$ are found to be

$$\mathbf{a}_z \cdot \mathbf{a}_r = \cos\theta$$
$$\mathbf{a}_z \cdot \mathbf{a}_\theta = -\sin\theta$$
$$\mathbf{a}_z \cdot \mathbf{a}_\phi = 0$$

The dot products involving $\mathbf{a}_x$ and $\mathbf{a}_y$ require first the projection of the spherical unit vector on the xy plane and then the projection onto the desired axis. For example, $\mathbf{a}_r \cdot \mathbf{a}_x$ is obtained by projecting $\mathbf{a}_r$ onto the xy plane, giving $\sin\theta$, and then projecting $\sin\theta$ on the x axis, which yields $\sin\theta\cos\phi$. The other dot products are found in a like manner, and all are shown in Table 1.2.

EXAMPLE 1.4

We illustrate this procedure by transforming the vector field $\mathbf{G} = (xz/y)\mathbf{a}_x$ into spherical components and variables.

Solution. We find the three spherical components by dotting $\mathbf{G}$ with the appropriate unit vectors, and we change variables during the procedure:

$$G_r = \mathbf{G} \cdot \mathbf{a}_r = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_r = \frac{xz}{y}\sin\theta\cos\phi$$
$$= r\sin\theta\cos\theta\frac{\cos^2\phi}{\sin\phi}$$
$$G_\theta = \mathbf{G} \cdot \mathbf{a}_\theta = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_\theta = \frac{xz}{y}\cos\theta\cos\phi$$
$$= r\cos^2\theta\frac{\cos^2\phi}{\sin\phi}$$
$$G\phi = \mathbf{G} \cdot \mathbf{a}_\phi = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_\phi = \frac{xz}{y}(-\sin\phi)$$
$$= -r\cos\theta\cos\phi$$

Collecting these results, we have

$$\mathbf{G} = r\cos\theta\cos\phi\,(\sin\theta\cot\phi\,\mathbf{a}_r + \cos\theta\cot\phi\,\mathbf{a}_\theta - \mathbf{a}_\phi)$$

Appendix A describes the general curvilinear coordinate system of which the rectangular, circular cylindrical, and spherical coordinate systems are special cases. The first section of this appendix could well be scanned now.

D1.7. Given the two points, $C(-3, 2, 1)$ and $D(r = 5, \theta = 20^\circ, \phi = -70^\circ)$, find: (*a*) the spherical coordinates of C; (*b*) the rectangular coordinates of D; (*c*) the distance from C to D.

Ans. $C(r = 3.74, \theta = 74.5^\circ, \phi = 146.3^\circ)$; $D(x = 0.585, y = -1.607, z = 4.70)$; 6.29

D1.8. Transform the following vectors to spherical coordinates at the points given: (*a*) $10\mathbf{a}_x$ at $P(x = -3, y = 2, z = 4)$; (*b*) $10\mathbf{a}_y$ at $Q(\rho = 5, \phi = 30^\circ, z = 4)$; (*c*) $10\mathbf{a}_z$ at $M(r = 4, \theta = 110^\circ, \phi = 120^\circ)$.

Ans. $-5.57\mathbf{a}_r - 6.18\mathbf{a}_\theta - 5.55\mathbf{a}_\phi$; $3.90\mathbf{a}_r + 3.12\mathbf{a}_\theta + 8.66\mathbf{a}_\phi$; $-3.42\mathbf{a}_r - 9.40\mathbf{a}_\theta$

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CHAPTER 1 PROBLEMS

WWW Quizzes

1.1 Given the vectors $\mathbf{M} = -10\mathbf{a}_x + 4\mathbf{a}_y - 8\mathbf{a}_z$ and $\mathbf{N} = 8\mathbf{a}_x + 7\mathbf{a}_y - 2\mathbf{a}_z$, find: (*a*) a unit vector in the direction of $-\mathbf{M} + 2\mathbf{N}$; (*b*) the magnitude of $5\mathbf{a}_x + \mathbf{N} - 3\mathbf{M}$; (*c*) $|\mathbf{M}||2\mathbf{N}|(\mathbf{M} + \mathbf{N})$.

1.2 Vector **A** extends from the origin to (1, 2, 3), and vector **B** extends from the origin to (2, 3, −2). Find (*a*) the unit vector in the direction of $(\mathbf{A} - \mathbf{B})$; (*b*) the unit vector in the direction of the line extending from the origin to the midpoint of the line joining the ends of **A** and **B**.

1.3 The vector from the origin to point A is given as (6, −2, −4), and the unit vector directed from the origin toward point B is (2, −2, 1)/3. If points A and B are ten units apart, find the coordinates of point B.

1.4 A circle, centered at the origin with a radius of 2 units, lies in the xy plane. Determine the unit vector in rectangular components that lies in the xy plane, is tangent to the circle at $(-\sqrt{3}, 1, 0)$, and is in the general direction of increasing values of y.

1.5 A vector field is specified as $\mathbf{G} = 24xy\mathbf{a}_x + 12(x^2 + 2)\mathbf{a}_y + 18z^2\mathbf{a}_z$. Given two points, $P(1, 2, -1)$ and $Q(-2, 1, 3)$, find (*a*) $\mathbf{G}$ at P; (*b*) a unit vector in the direction of $\mathbf{G}$ at Q; (*c*) a unit vector directed from Q toward P; (*d*) the equation of the surface on which $|\mathbf{G}| = 60$.

1.6 Find the acute angle between the two vectors $\mathbf{A} = 2\mathbf{a}_x + \mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{B} = \mathbf{a}_x - 3\mathbf{a}_y + 2\mathbf{a}_z$ by using the definition of (*a*) the dot product; (*b*) the cross product.

1.7 Given the vector field $\mathbf{E} = 4zy^2 \cos 2x\mathbf{a}_x + 2zy \sin 2x\mathbf{a}_y + y^2 \sin 2x\mathbf{a}_z$ for the region $|x|$, $|y|$, and $|z|$ less than 2, find (*a*) the surfaces on which $E_y = 0$; (*b*) the region in which $E_y = E_z$; (*c*) the region in which $\mathbf{E} = 0$.

1.8 Demonstrate the ambiguity that results when the cross product is used to find the angle between two vectors by finding the angle between $\mathbf{A} = 3\mathbf{a}_x - 2\mathbf{a}_y + 4\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z$. Does this ambiguity exist when the dot product is used?

1.9 A field is given as $\mathbf{G} = [25/(x^2 + y^2)](x\mathbf{a}_x + y\mathbf{a}_y)$. Find (*a*) a unit vector in the direction of $\mathbf{G}$ at $P(3, 4, -2)$; (*b*) the angle between $\mathbf{G}$ and $\mathbf{a}_x$ at P; (*c*) the value of the following double integral on the plane $y = 7$.

$$\int_0^4 \int_0^2 \mathbf{G} \cdot \mathbf{a}_y \, dz dx$$

1.10 By expressing diagonals as vectors and using the definition of the dot product, find the smaller angle between any two diagonals of a cube, where each diagonal connects diametrically opposite corners and passes through the center of the cube.

1.11 Given the points $M(0.1, -0.2, -0.1)$, $N(-0.2, 0.1, 0.3)$, and $P(0.4, 0, 0.1)$, find (*a*) the vector $\mathbf{R}_{MN}$; (*b*) the dot product $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}$; (*c*) the scalar projection of $\mathbf{R}_{MN}$ on $\mathbf{R}_{MP}$; (*d*) the angle between $\mathbf{R}_{MN}$ and $\mathbf{R}_{MP}$.

1.12 Write an expression in rectangular components for the vector that extends from (x_1, y_1, z_1) to (x_2, y_2, z_2) and determine the magnitude of this vector.

1.13 Find (*a*) the vector component of $\mathbf{F} = 10\mathbf{a}_x - 6\mathbf{a}_y + 5\mathbf{a}_z$ that is parallel to $\mathbf{G} = 0.1\mathbf{a}_x + 0.2\mathbf{a}_y + 0.3\mathbf{a}_z$; (*b*) the vector component of $\mathbf{F}$ that is perpendicular to $\mathbf{G}$; (*c*) the vector component of $\mathbf{G}$ that is perpendicular to $\mathbf{F}$.

1.14 Given that $\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$, where the three vectors represent line segments and extend from a common origin, must the three vectors be coplanar? If $\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 0$, are the four vectors coplanar?

1.15 Three vectors extending from the origin are given as $\mathbf{r}_1 = (7, 3, -2)$, $\mathbf{r}_2 = (-2, 7, -3)$, and $\mathbf{r}_3 = (0, 2, 3)$. Find (*a*) a unit vector perpendicular to both $\mathbf{r}_1$ and $\mathbf{r}_2$; (*b*) a unit vector perpendicular to the vectors $\mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{r}_2 - \mathbf{r}_3$; (*c*) the area of the triangle defined by $\mathbf{r}_1$ and $\mathbf{r}_2$; (*d*) the area of the triangle defined by the heads of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$.

1.16 If **A** represents a vector one unit long directed due east, **B** represents a vector three units long directed due north, and $\mathbf{A} + \mathbf{B} = 2\mathbf{C} - \mathbf{D}$ and $2\mathbf{A} - \mathbf{B} = \mathbf{C} + 2\mathbf{D}$, determine the length and direction of **C**.

1.17 Point $A(-4, 2, 5)$ and the two vectors, $\mathbf{R}_{AM} = (20, 18 - 10)$ and $\mathbf{R}_{AN} = (-10, 8, 15)$, define a triangle. Find (*a*) a unit vector perpendicular to the triangle; (*b*) a unit vector in the plane of the triangle and perpendicular to $\mathbf{R}_{AN}$; (*c*) a unit vector in the plane of the triangle that bisects the interior angle at A.

1.18 A certain vector field is given as $\mathbf{G} = (y + 1)\mathbf{a}_x + x\mathbf{a}_y$. (*a*) Determine **G** at the point $(3, -2, 4)$; (*b*) obtain a unit vector defining the direction of **G** at $(3, -2, 4)$.

1.19 (*a*) Express the field $\mathbf{D} = (x^2 + y^2)^{-1}(x\mathbf{a}_x + y\mathbf{a}_y)$ in cylindrical components and cylindrical variables. (*b*) Evaluate **D** at the point where $\rho = 2$, $\phi = 0.2\pi$, and $z = 5$, expressing the result in cylindrical and rectangular components.

1.20 If the three sides of a triangle are represented by vectors **A**, **B**, and **C**, all directed counterclockwise, show that $|\mathbf{C}|^2 = (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B})$ and expand the product to obtain the law of cosines.

1.21 Express in cylindrical components: (*a*) the vector from $C(3, 2, -7)$ to $D(-1, -4, 2)$; (*b*) a unit vector at D directed toward C; (*c*) a unit vector at D directed toward the origin.

1.22 A sphere of radius a, centered at the origin, rotates about the z axis at angular velocity Ω rad/s. The rotation direction is clockwise when one is looking in the positive z direction. (*a*) Using spherical components, write an expression for the velocity field, **v**, that gives the tangential velocity at any point within the sphere; (*b*) convert to rectangular components.

1.23 The surfaces $\rho = 3$, $\rho = 5$, $\phi = 100^\circ$, $\phi = 130^\circ$, $z = 3$, and $z = 4.5$ define a closed surface. Find (*a*) the enclosed volume; (*b*) the total area of the enclosing surface; (*c*) the total length of the twelve edges of the surfaces; (*d*) the length of the longest straight line that lies entirely within the volume.

1.24 Two unit vectors, $\mathbf{a}_1$ and $\mathbf{a}_2$, lie in the xy plane and pass through the origin. They make angles ϕ_1 and ϕ_2, respectively, with the x axis (*a*) Express each vector in rectangular components; (*b*) take the dot product and verify the trigonometric identity, $\cos(\phi_1 - \phi_2) = \cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2$; (*c*) take the cross product and verify the trigonometric identity $\sin(\phi_2 - \phi_1) = \sin\phi_2 \cos\phi_1 - \cos\phi_2 \sin\phi_1$.

1.25 Given point $P(r = 0.8, \theta = 30^\circ, \phi = 45^\circ)$ and $\mathbf{E} = 1/r^2\ [\cos\phi\,\mathbf{a}_r + (\sin\phi/\sin\theta)\,\mathbf{a}_\phi]$, find (*a*) $\mathbf{E}$ at P; (*b*) $|\mathbf{E}|$ at P; (*c*) a unit vector in the direction of $\mathbf{E}$ at P.

1.26 Express the uniform vector field $\mathbf{F} = 5\mathbf{a}_x$ in (*a*) cylindrical components; (*b*) spherical components.

1.27 The surfaces $r = 2$ and 4, $\theta = 30^\circ$ and 50°, and $\phi = 20^\circ$ and 60° identify a closed surface. Find (*a*) the enclosed volume; (*b*) the total area of the enclosing surface; (*c*) the total length of the twelve edges of the surface; (*d*) the length of the longest straight line that lies entirely within the surface.

1.28 State whether or not $\mathbf{A} = \mathbf{B}$ and, if not, what conditions are imposed on $\mathbf{A}$ and $\mathbf{B}$ when (*a*) $\mathbf{A}\cdot\mathbf{a}_x = \mathbf{B}\cdot\mathbf{a}_x$; (*b*) $\mathbf{A}\times\mathbf{a}_x = \mathbf{B}\times\mathbf{a}_x$; (*c*) $\mathbf{A}\cdot\mathbf{a}_x = \mathbf{B}\cdot\mathbf{a}_x$ *and* $\mathbf{A}\times\mathbf{a}_x = \mathbf{B}\times\mathbf{a}_x$; (*d*) $\mathbf{A}\cdot\mathbf{C} = \mathbf{B}\cdot\mathbf{C}$ *and* $\mathbf{A}\times\mathbf{C} = \mathbf{B}\times\mathbf{C}$ where $\mathbf{C}$ is any vector except $\mathbf{C} = 0$.

1.29 Express the unit vector $\mathbf{a}_x$ in spherical components at the point: (*a*) $r = 2$, $\theta = 1$ rad, $\phi = 0.8$ rad; (*b*) $x = 3$, $y = 2$, $z = -1$; (*c*) $\rho = 2.5$, $\phi = 0.7$ rad, $z = 1.5$.

1.30 Consider a problem analogous to the varying wind velocities encountered by transcontinental aircraft. We assume a constant altitude, a plane earth, a flight along the x axis from 0 to 10 units, no vertical velocity component, and no change in wind velocity with time. Assume $\mathbf{a}_x$ to be directed to the east and $\mathbf{a}_y$ to the north. The wind velocity at the operating altitude is assumed to be:

$$\mathbf{v}(x, y) = \frac{(0.01x^2 - 0.08x + 0.66)\mathbf{a}_x - (0.05x - 0.4)\mathbf{a}_y}{1 + 0.5y^2}$$

Determine the location and magnitude of (*a*) the maximum tailwind encountered; (*b*) repeat for headwind; (*c*) repeat for crosswind; (*d*) Would more favorable tailwinds be available at some other latitude? If so, where?

2 CHAPTER

Coulomb's Law and Electric Field Intensity

Having formulated the language of vector analysis in the first chapter, we next establish and describe a few basic principles of electricity. In this chapter, we introduce Coulomb's electrostatic force law and then formulate this in a general way using field theory. The tools that will be developed can be used to solve any problem in which forces between static charges are to be evaluated or to determine the electric field that is associated with any charge distribution. Initially, we will restrict the study to fields in *vacuum* or *free space*; this would apply to media such as air and other gases. Other materials are introduced in Chapters 5 and 6 and time-varying fields are introduced in Chapter 9. ■

2.1 THE EXPERIMENTAL LAW OF COULOMB

Records from at least 600 B.C. show evidence of the knowledge of static electricity. The Greeks were responsible for the term *electricity*, derived from their word for amber, and they spent many leisure hours rubbing a small piece of amber on their sleeves and observing how it would then attract pieces of fluff and stuff. However, their main interest lay in philosophy and logic, not in experimental science, and it was many centuries before the attracting effect was considered to be anything other than magic or a "life force."

Dr. Gilbert, physician to Her Majesty the Queen of England, was the first to do any true experimental work with this effect, and in 1600 he stated that glass, sulfur, amber, and other materials, which he named, would "not only draw to themselves straws and chaff, but all metals, wood, leaves, stone, earths, even water and oil."

Shortly thereafter, an officer in the French Army Engineers, Colonel Charles Coulomb, performed an elaborate series of experiments using a delicate torsion balance, invented by himself, to determine quantitatively the force exerted between two objects, each having a static charge of electricity. His published result bears a great similarity to Newton's gravitational law (discovered about a hundred years earlier).

Coulomb stated that the force between two very small objects separated in a vacuum or free space by a distance, which is large compared to their size, is proportional to the charge on each and inversely proportional to the square of the distance between them, or

$$F = k\frac{Q_1 Q_2}{R^2}$$

where Q_1 and Q_2 are the positive or negative quantities of charge, R is the separation, and k is a proportionality constant. If the International System of Units[1] (SI) is used, Q is measured in coulombs (C), R is in meters (m), and the force should be newtons (N). This will be achieved if the constant of proportionality k is written as

$$k = \frac{1}{4\pi\epsilon_0}$$

The new constant ϵ_0 is called the *permittivity of free space* and has magnitude, measured in farads per meter (F/m),

$$\epsilon_0 = 8.854 \times 10^{-12} \doteq \frac{1}{36\pi} 10^{-9} \text{ F/m} \tag{1}$$

The quantity ϵ_0 is not dimensionless, for Coulomb's law shows that it has the label $C^2/N \cdot m^2$. We will later define the farad and show that it has the dimensions $C^2/N \cdot m$; we have anticipated this definition by using the unit F/m in equation (1).

Coulomb's law is now

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \tag{2}$$

The coulomb is an extremely large unit of charge, for the smallest known quantity of charge is that of the electron (negative) or proton (positive), given in SI units as 1.602×10^{-19} C; hence a negative charge of one coulomb represents about 6×10^{18} electrons.[2] Coulomb's law shows that the force between two charges of one coulomb each, separated by one meter, is 9×10^9 N, or about one million tons. The electron has a rest mass of 9.109×10^{-31}kg and has a radius of the order of magnitude of 3.8×10^{-15} m. This does not mean that the electron is spherical in shape, but merely describes the size of the region in which a slowly moving electron has the greatest probability of being found. All other known charged particles, including the proton, have larger masses and larger radii, and occupy a probabilistic volume larger than does the electron.

In order to write the vector form of (2), we need the additional fact (furnished also by Colonel Coulomb) that the force acts along the line joining the two charges

[1] The International System of Units (an mks system) is described in Appendix B. Abbreviations for the units are given in Table B.1. Conversions to other systems of units are given in Table B.2, while the prefixes designating powers of ten in SI appear in Table B.3.

[2] The charge and mass of an electron and other physical constants are tabulated in Table C.4 of Appendix C.

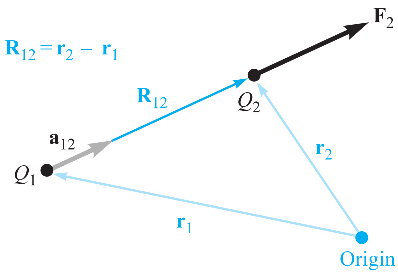

Figure 2.1 If Q_1 and Q_2 have like signs, the vector force $\mathbf{F}_2$ on Q_2 is in the same direction as the vector $\mathbf{R}_{12}$.

and is repulsive if the charges are alike in sign or attractive if they are of opposite sign. Let the vector $\mathbf{r}_1$ locate Q_1, whereas $\mathbf{r}_2$ locates Q_2. Then the vector $\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ represents the directed line segment from Q_1 to Q_2, as shown in Figure 2.1. The vector $\mathbf{F}_2$ is the force on Q_2 and is shown for the case where Q_1 and Q_2 have the same sign. The vector form of Coulomb's law is

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12} \tag{3}$$

where $\mathbf{a}_{12}$ = a unit vector in the direction of R_{12}, or

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \tag{4}$$

EXAMPLE 2.1

We illustrate the use of the vector form of Coulomb's law by locating a charge of $Q_1 = 3 \times 10^{-4}$ C at $M(1, 2, 3)$ and a charge of $Q_2 = -10^{-4}$ C at $N(2, 0, 5)$ in a vacuum. We desire the force exerted on Q_2 by Q_1.

Solution. We use (3) and (4) to obtain the vector force. The vector $\mathbf{R}_{12}$ is

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1 = (2-1)\mathbf{a}_x + (0-2)\mathbf{a}_y + (5-3)\mathbf{a}_z = \mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z$$

leading to $|\mathbf{R}_{12}| = 3$, and the unit vector, $\mathbf{a}_{12} = \frac{1}{3}(\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)$. Thus,

$$\mathbf{F}_2 = \frac{3 \times 10^{-4}(-10^{-4})}{4\pi(1/36\pi)10^{-9} \times 3^2}\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right)$$
$$= -30\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right) \text{ N}$$

The magnitude of the force is 30 N, and the direction is specified by the unit vector, which has been left in parentheses to display the magnitude of the force. The force on Q_2 may also be considered as three component forces,

$$\mathbf{F}_2 = -10\mathbf{a}_x + 20\mathbf{a}_y - 20\mathbf{a}_z$$

The force expressed by Coulomb's law is a mutual force, for each of the two charges experiences a force of the same magnitude, although of opposite direction. We might equally well have written

$$\mathbf{F}_1 = -\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{21} = -\frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12} \tag{5}$$

Coulomb's law is linear, for if we multiply Q_1 by a factor n, the force on Q_2 is also multiplied by the same factor n. It is also true that the force on a charge in the presence of several other charges is the sum of the forces on that charge due to each of the other charges acting alone.

D2.1. A charge $Q_A = -20\,\mu$C is located at $A(-6, 4, 7)$, and a charge $Q_B = 50\,\mu$C is at $B(5, 8, -2)$ in free space. If distances are given in meters, find: (*a*) $\mathbf{R}_{AB}$; (*b*) R_{AB}. Determine the vector force exerted on Q_A by Q_B if $\epsilon_0 =$ (*c*) $10^{-9}/(36\pi)$ F/m; (*d*) 8.854×10^{-12} F/m.

Ans. $11\mathbf{a}_x + 4\mathbf{a}_y - 9\mathbf{a}_z$ m; 14.76 m; $30.76\mathbf{a}_x + 11.184\mathbf{a}_y - 25.16\mathbf{a}_z$ mN; $30.72\mathbf{a}_x + 11.169\mathbf{a}_y - 25.13\mathbf{a}_z$ mN

2.2 ELECTRIC FIELD INTENSITY

If we now consider one charge fixed in position, say Q_1, and move a second charge slowly around, we note that there exists everywhere a force on this second charge; in other words, this second charge is displaying the existence of a force *field* that is associated with charge, Q_1. Call this second charge a test charge Q_t. The force on it is given by Coulomb's law,

$$\mathbf{F}_t = \frac{Q_1 Q_t}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t}$$

Writing this force as a force per unit charge gives the *electric field intensity*, $\mathbf{E}_1$ arising from Q_1:

$$\mathbf{E}_1 = \frac{\mathbf{F}_t}{Q_1} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t} \tag{6}$$

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$\mathbf{E}_1$ is interpreted as the vector force, arising from charge Q_1, that acts on a unit positive test charge. More generally, we write the defining expression:

$$\mathbf{E} = \frac{\mathbf{F}_t}{Q_t} \tag{7}$$

in which $\mathbf{E}$, a vector function, is the electric field intensity *evaluated at the test charge location* that arises from all *other* charges in the vicinity—meaning the electric field arising from the test charge itself is not included in $\mathbf{E}$.

The units of $\mathbf{E}$ would be in force per unit charge (newtons per coulomb). Again anticipating a new dimensional quantity, the *volt* (V), having the label of joules per coulomb (J/C), or newton-meters per coulomb ($N \cdot m/C$), we measure electric field intensity in the practical units of volts per meter (V/m).

Now, we dispense with most of the subscripts in (6), reserving the right to use them again any time there is a possibility of misunderstanding. The electric field of a single point charge becomes:

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 R^2}\mathbf{a}_R \tag{8}$$

We remember that R is the magnitude of the vector $\mathbf{R}$, the directed line segment from the point at which the point charge Q is located to the point at which $\mathbf{E}$ is desired, and $\mathbf{a}_R$ is a unit vector in the $\mathbf{R}$ direction.[3]

We arbitrarily locate Q_1 at the center of a spherical coordinate system. The unit vector $\mathbf{a}_R$ then becomes the radial unit vector $\mathbf{a}_r$, and R is r. Hence

$$\mathbf{E} = \frac{Q_1}{4\pi\epsilon_0 r^2}\mathbf{a}_r \tag{9}$$

The field has a single radial component, and its inverse-square-law relationship is quite obvious.

If we consider a charge that is *not* at the origin of our coordinate system, the field no longer possesses spherical symmetry, and we might as well use rectangular coordinates. For a charge Q located at the source point $\mathbf{r}' = x'\mathbf{a}_x + y'\mathbf{a}_y + z'\mathbf{a}_z$, as illustrated in Figure 2.2, we find the field at a general field point $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ by expressing $\mathbf{R}$ as $\mathbf{r} - \mathbf{r}'$, and then

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{Q(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$

$$= \frac{Q[(x-x')\mathbf{a}_x + (y-y')\mathbf{a}_y + (z-z')\mathbf{a}_z]}{4\pi\epsilon_0[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}} \tag{10}$$

Earlier, we defined a vector field as a vector function of a position vector, and this is emphasized by letting $\mathbf{E}$ be symbolized in functional notation by $\mathbf{E}(\mathbf{r})$.

Because the coulomb forces are linear, the electric field intensity arising from two point charges, Q_1 at $\mathbf{r}_1$ and Q_2 at $\mathbf{r}_2$, is the sum of the forces on Q_t caused by Q_1 and Q_2 acting alone, or

$$\mathbf{E}(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are unit vectors in the direction of $(\mathbf{r}-\mathbf{r}_1)$ and $(\mathbf{r}-\mathbf{r}_2)$, respectively. The vectors $\mathbf{r}$, $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}-\mathbf{r}_1$, $\mathbf{r}-\mathbf{r}_2$, $\mathbf{a}_1$, and $\mathbf{a}_2$ are shown in Figure 2.3.

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[3] We firmly intend to avoid confusing r and $\mathbf{a}_r$ with R and $\mathbf{a}_R$. The first two refer specifically to the spherical coordinate system, whereas R and $\mathbf{a}_R$ do not refer to any coordinate system—the choice is still available to us.

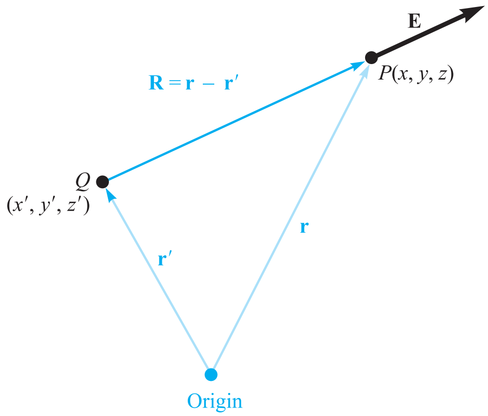

Figure 2.2 The vector $\mathbf{r}'$ locates the point charge Q, the vector $\mathbf{r}$ identifies the general point in space $P(x, y, z)$, and the vector $\mathbf{R}$ from Q to $P(x, y, z)$ is then $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

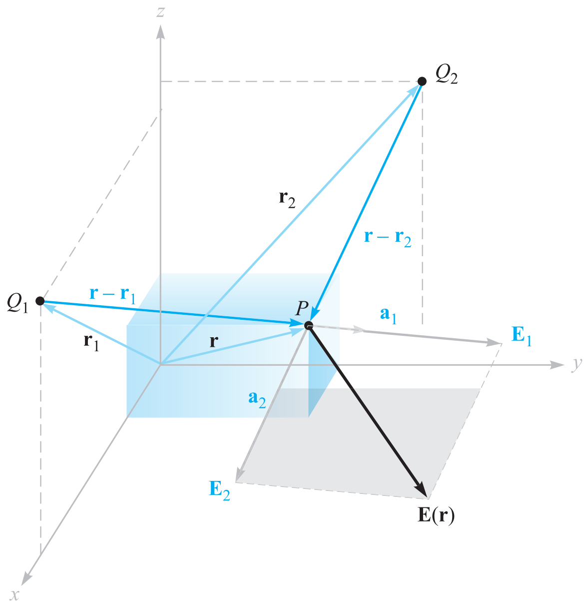

Figure 2.3 The vector addition of the total electric field intensity at P due to Q_1 and Q_2 is made possible by the linearity of Coulomb's law.

If we add more charges at other positions, the field due to n point charges is

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$$\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_m|^2} \mathbf{a}_m \tag{11}$$

EXAMPLE 2.2

In order to illustrate the application of (11), we find $\mathbf{E}$ at $P(1, 1, 1)$ caused by four identical 3-nC (nanocoulomb) charges located at $P_1(1, 1, 0)$, $P_2(-1, 1, 0)$, $P_3(-1, -1, 0)$, and $P_4(1, -1, 0)$, as shown in Figure 2.4.

Solution. We find that $\mathbf{r} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$, $\mathbf{r}_1 = \mathbf{a}_x + \mathbf{a}_y$, and thus $\mathbf{r} - \mathbf{r}_1 = \mathbf{a}_z$. The magnitudes are: $|\mathbf{r} - \mathbf{r}_1| = 1$, $|\mathbf{r} - \mathbf{r}_2| = \sqrt{5}$, $|\mathbf{r} - \mathbf{r}_3| = 3$, and $|\mathbf{r} - \mathbf{r}_4| = \sqrt{5}$. Because $Q/4\pi\epsilon_0 = 3 \times 10^{-9}/(4\pi \times 8.854 \times 10^{-12}) = 26.96$ V · m, we may now use (11) to obtain

$$\mathbf{E} = 26.96 \left[\frac{\mathbf{a}_z}{1} \frac{1}{1^2} + \frac{2\mathbf{a}_x + \mathbf{a}_z}{\sqrt{5}} \frac{1}{(\sqrt{5})^2} + \frac{2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z}{3} \frac{1}{3^2} + \frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}} \frac{1}{(\sqrt{5})^2} \right]$$

or

$$\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z \text{ V/m}$$

D2.2. A charge of $-0.3\,\mu$C is located at $A(25, -30, 15)$ (in cm), and a second charge of $0.5\,\mu$C is at $B(-10, 8, 12)$ cm. Find $\mathbf{E}$ at: (*a*) the origin; (*b*) $P(15, 20, 50)$ cm.

Ans. $92.3\mathbf{a}_x - 77.6\mathbf{a}_y - 94.2\mathbf{a}_z$ kV/m; $11.9\mathbf{a}_x - 0.519\mathbf{a}_y + 12.4\mathbf{a}_z$ kV/m

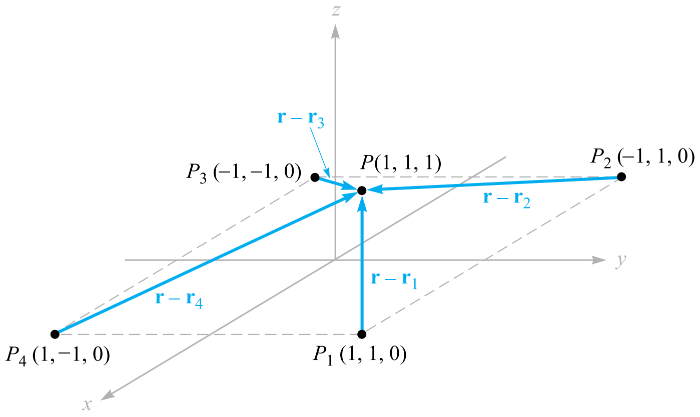

Figure 2.4 A symmetrical distribution of four identical 3-nC point charges produces a field at P, $\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z$ V/m.

D2.3. Evaluate the sums: (*a*) $\sum_{m=0}^{5} \frac{1+(-1)^m}{m^2+1}$; (*b*) $\sum_{m=1}^{4} \frac{(0.1)^m+1}{(4+m^2)^{1.5}}$

Ans. 2.52; 0.176

2.3 FIELD ARISING FROM A CONTINUOUS VOLUME CHARGE DISTRIBUTION

WWW Illustrations

If we now visualize a region of space filled with a tremendous number of charges separated by minute distances, we see that we can replace this distribution of very small particles with a smooth continuous distribution described by a *volume charge density*, just as we describe water as having a density of 1 g/cm^3 (gram per cubic centimeter) even though it consists of atomic- and molecular-sized particles. We can do this only if we are uninterested in the small irregularities (or ripples) in the field as we move from electron to electron or if we care little that the mass of the water actually increases in small but finite steps as each new molecule is added.

This is really no limitation at all, because the end results for electrical engineers are almost always in terms of a current in a receiving antenna, a voltage in an electronic circuit, or a charge on a capacitor, or in general in terms of some large-scale *macroscopic* phenomenon. It is very seldom that we must know a current electron by electron.[4]

We denote volume charge density by ρ_ν, having the units of coulombs per cubic meter (C/m^3).

The small amount of charge ΔQ in a small volume $\Delta \nu$ is

$$\Delta Q = \rho_\nu \Delta \nu \tag{12}$$

and we may define ρ_ν mathematically by using a limiting process on (12),

$$\rho_\nu = \lim_{\Delta \nu \to 0} \frac{\Delta Q}{\Delta \nu} \tag{13}$$

The total charge within some finite volume is obtained by integrating throughout that volume,

$$Q = \int_{\text{vol}} \rho_\nu d\nu \tag{14}$$

Only one integral sign is customarily indicated, but the differential $d\nu$ signifies integration throughout a volume, and hence a triple integration.

[4] A study of the noise generated by electrons in semiconductors and resistors, however, requires just such an examination of the charge through statistical analysis.

EXAMPLE 2.3

As an example of the evaluation of a volume integral, we find the total charge contained in a 2-cm length of the electron beam shown in Figure 2.5.

Solution. From the illustration, we see that the charge density is

$$\rho_v = -5 \times 10^{-6} e^{-10^5 \rho z} \text{ C/m}^2$$

The volume differential in cylindrical coordinates is given in Section 1.8; therefore,

$$Q = \int_{0.02}^{0.04} \int_0^{2\pi} \int_0^{0.01} -5 \times 10^{-6} e^{-10^5 \rho z} \rho \, d\rho \, d\phi \, dz$$

We integrate first with respect to ϕ because it is so easy,

$$Q = \int_{0.02}^{0.04} \int_0^{0.01} -10^{-5} \pi e^{-10^5 \rho z} \rho \, d\rho \, dz$$

and then with respect to z, because this will simplify the last integration with respect to ρ,

$$Q = \int_0^{0.01} \left(\frac{-10^{-5}\pi}{-10^5 \rho} e^{-10^5 \rho z} \rho \, d\rho \right)_{z=0.02}^{z=0.04}$$

$$= \int_0^{0.01} -10^{-5} \pi (e^{-2000\rho} - e^{-4000\rho}) d\rho$$

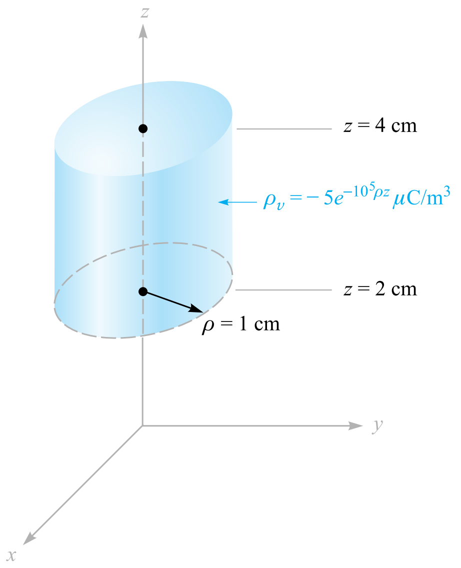

Figure 2.5 The total charge contained within the right circular cylinder may be obtained by evaluating $Q = \int_{\text{vol}} \rho_v dv$.

Finally,

$$Q = -10^{-10}\pi \left(\frac{e^{-2000\rho}}{-2000} - \frac{e^{-4000\rho}}{-4000} \right)_0^{0.01}$$

$$Q = -10^{-10}\pi \left(\frac{1}{2000} - \frac{1}{4000} \right) = \frac{-\pi}{40} = 0.0785 \text{ pC}$$

where pC indicates picocoulombs.

The incremental contribution to the electric field intensity at $\mathbf{r}$ produced by an incremental charge ΔQ at $\mathbf{r}'$ is

$$\Delta\mathbf{E}(\mathbf{r}) = \frac{\Delta Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{\rho_\nu \Delta\nu}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

If we sum the contributions of all the volume charge in a given region and let the volume element $\Delta\nu$ approach zero as the number of these elements becomes infinite, the summation becomes an integral,

$$\mathbf{E}(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_\nu(\mathbf{r}')\,d\nu'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \qquad (15)$$

This is again a triple integral, and (except in Drill Problem 2.4) we shall do our best to avoid actually performing the integration.

The significance of the various quantities under the integral sign of (15) might stand a little review. The vector $\mathbf{r}$ from the origin locates the field point where $\mathbf{E}$ is being determined, whereas the vector $\mathbf{r}'$ extends from the origin to the source point where $\rho_\nu(\mathbf{r}')d\nu'$ is located. The scalar distance between the source point and the field point is $|\mathbf{r}-\mathbf{r}'|$, and the fraction $(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|$ is a unit vector directed from source point to field point. The variables of integration are x', y', and z' in rectangular coordinates.

D2.4. Calculate the total charge within each of the indicated volumes: (*a*) $0.1 \le |x|, |y|, |z| \le 0.2$: $\rho_\nu = \dfrac{1}{x^3y^3z^3}$; (*b*) $0 \le \rho \le 0.1, 0 \le \phi \le \pi, 2 \le z \le 4$; $\rho_\nu = \rho^2 z^2 \sin 0.6\phi$; (*c*) universe: $\rho_\nu = e^{-2r}/r^2$.

Ans. 0; 1.018 mC; 6.28 C

2.4 FIELD OF A LINE CHARGE

Up to this point we have considered two types of charge distribution, the point charge and charge distributed throughout a volume with a density ρ_ν C/m^3. If we now consider a filamentlike distribution of volume charge density, such as a charged conductor of very small radius, we find it convenient to treat the charge as a line charge of density ρ_L C/m.

We assume a straight-line charge extending along the z axis in a cylindrical coordinate system from $-\infty$ to ∞, as shown in Figure 2.6. We desire the electric field intensity $\mathbf{E}$ at any and every point resulting from a *uniform* line charge density ρ_L.

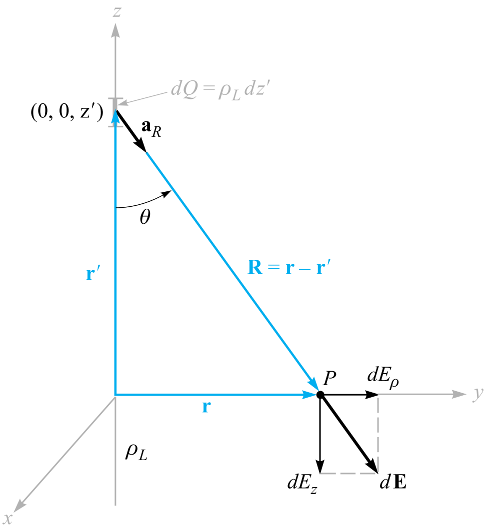

Figure 2.6 The contribution $d\mathbf{E} = dE_\rho \mathbf{a}_\rho + dE_z \mathbf{a}_z$ to the electric field intensity produced by an element of charge $dQ = \rho_L dz'$ located a distance z' from the origin. The linear charge density is uniform and extends along the entire z axis.

Symmetry should always be considered first in order to determine two specific factors: (1) with which coordinates the field does *not* vary, and (2) which components of the field are *not* present. The answers to these questions then tell us which components are present and with which coordinates they *do* vary.

Referring to Figure 2.6, we realize that as we move around the line charge, varying ϕ while keeping ρ and z constant, the line charge appears the same from every angle. In other words, azimuthal symmetry is present, and no field component may vary with ϕ.

Again, if we maintain ρ and ϕ constant while moving up and down the line charge by changing z, the line charge still recedes into infinite distance in both directions and the problem is unchanged. This is axial symmetry and leads to fields that are not functions of z.

If we maintain ϕ and z constant and vary ρ, the problem changes, and Coulomb's law leads us to expect the field to become weaker as ρ increases. Hence, by a process of elimination we are led to the fact that the field varies only with ρ.

Now, which components are present? Each incremental length of line charge acts as a point charge and produces an incremental contribution to the electric field intensity which is directed away from the bit of charge (assuming a positive line charge). No element of charge produces a ϕ component of electric intensity; E_ϕ is zero. However, each element does produce an E_ρ and an E_z component, but the contribution to E_z by elements of charge that are equal distances above and below the point at which we are determining the field will cancel.

We therefore have found that we have only an E_ρ component and it varies only with ρ. Now to find this component.

We choose a point $P(0, y, 0)$ on the y axis at which to determine the field. This is a perfectly general point in view of the lack of variation of the field with ϕ and z. Applying (10) to find the incremental field at P due to the incremental charge $dQ = \rho_L dz'$, we have

$$d\mathbf{E} = \frac{\rho_L dz'(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where

$$\mathbf{r} = y\mathbf{a}_y = \rho\mathbf{a}_\rho$$
$$\mathbf{r}' = z'\mathbf{a}_z$$

and

$$\mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z'\mathbf{a}_z$$

Therefore,

$$d\mathbf{E} = \frac{\rho_L dz'(\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

Because only the $\mathbf{E}_\rho$ component is present, we may simplify:

$$dE_\rho = \frac{\rho_L \rho dz'}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

and

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \rho dz'}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

Integrating by integral tables or change of variable, $z' = \rho \cot\theta$, we have

$$E_\rho = \frac{\rho_L}{4\pi\epsilon_0}\rho \left(\frac{1}{\rho^2} \frac{z'}{\sqrt{\rho^2 + z'^2}} \right)_{-\infty}^{\infty}$$

and

$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho}$$

or finally,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0 \rho}\mathbf{a}_\rho \qquad (16)$$

WW Illustrations

We note that the field falls off inversely with the distance to the charged line, as compared with the point charge, where the field decreased with the *square* of the distance. Moving ten times as far from a point charge leads to a field only 1 percent the previous strength, but moving ten times as far from a line charge only reduces the field to 10 percent of its former value. An analogy can be drawn with a source of

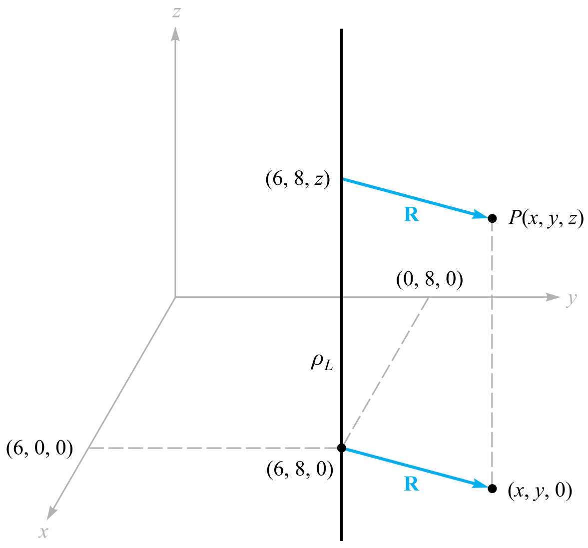

Figure 2.7 A point $P(x, y, z)$ is identified near an infinite uniform line charge located at $x = 6$, $y = 8$.

illumination, for the light intensity from a point source of light also falls off inversely as the square of the distance to the source. The field of an infinitely long fluorescent tube thus decays inversely as the first power of the radial distance to the tube, and we should expect the light intensity about a finite-length tube to obey this law near the tube. As our point recedes farther and farther from a finite-length tube, however, it eventually looks like a point source, and the field obeys the inverse-square relationship.

Before leaving this introductory look at the field of the infinite line charge, we should recognize the fact that not all line charges are located along the z axis. As an example, let us consider an infinite line charge parallel to the z axis at $x = 6$, $y = 8$, shown in Figure 2.7. We wish to find $\mathbf{E}$ at the general field point $P(x, y, z)$.

We replace ρ in (16) by the radial distance between the line charge and point, P, $R = \sqrt{(x-6)^2 + (y-8)^2}$, and let $\mathbf{a}_\rho$ be $\mathbf{a}_R$. Thus,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\sqrt{(x-6)^2 + (y-8)^2}}\mathbf{a}_R$$

where

$$\mathbf{a}_R = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{(x-6)\mathbf{a}_x + (y-8)\mathbf{a}_y}{\sqrt{(x-6)^2 + (y-8)^2}}$$

Therefore,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0}\frac{(x-6)\mathbf{a}_x + (y-8)\mathbf{a}_y}{(x-6)^2 + (y-8)^2}$$

We again note that the field is not a function of z.

In Section 2.6, we describe how fields may be sketched, and we use the field of the line charge as one example.

D2.5. Infinite uniform line charges of 5 nC/m lie along the (positive and negative) x and y axes in free space. Find $\mathbf{E}$ at: (a) $P_A(0, 0, 4)$; (b) $P_B(0, 3, 4)$.

Ans. $45\mathbf{a}_z$ V/m; $10.8\mathbf{a}_y + 36.9\mathbf{a}_z$ V/m

2.5 FIELD OF A SHEET OF CHARGE

Another basic charge configuration is the infinite sheet of charge having a uniform density of ρ_S C/m^2. Such a charge distribution may often be used to approximate that found on the conductors of a strip transmission line or a parallel-plate capacitor. As we shall see in Chapter 5, static charge resides on conductor surfaces and not in their interiors; for this reason, ρ_S is commonly known as *surface charge density*. The charge-distribution family now is complete—point, line, surface, and volume, or Q, ρ_L, ρ_S, and ρ_ν.

Let us place a sheet of charge in the yz plane and again consider symmetry (Figure 2.8). We see first that the field cannot vary with y or with z, and then we see that the y and z components arising from differential elements of charge symmetrically located with respect to the point at which we evaluate the field will cancel. Hence only E_x is present, and this component is a function of x alone. We are again faced with a choice of many methods by which to evaluate this component, and this time we use only one method and leave the others as exercises for a quiet Sunday afternoon.

Let us use the field of the infinite line charge (16) by dividing the infinite sheet into differential-width strips. One such strip is shown in Figure 2.8. The line charge

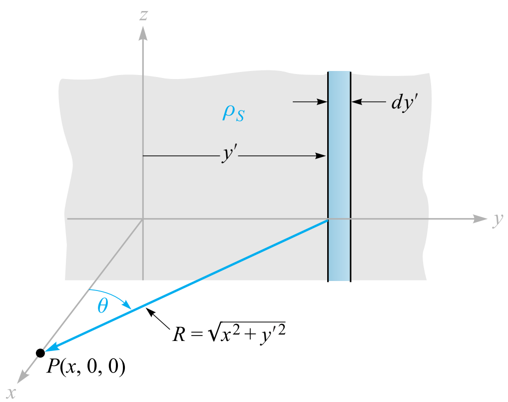

Figure 2.8 An infinite sheet of charge in the yz plane, a general point P on the x axis, and the differential-width line charge used as the element in determining the field at P by $d\mathbf{E} = \rho_S dy' \mathbf{a}_R/(2\pi\varepsilon_0 R)$.

density, or charge per unit length, is $\rho_L = \rho_S \, dy'$, and the distance from this line charge to our general point P on the x axis is $R = \sqrt{x^2 + y'^2}$. The contribution to E_x at P from this differential-width strip is then

$$dE_x = \frac{\rho_S \, dy'}{2\pi\epsilon_0 \sqrt{x^2 + y'^2}} \cos\theta = \frac{\rho_S}{2\pi\epsilon_0} \frac{x \, dy'}{x^2 + y'^2}$$

Adding the effects of all the strips,

$$E_x = \frac{\rho_S}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{x \, dy'}{x^2 + y'^2} = \frac{\rho_S}{2\pi\epsilon_0} \tan^{-1} \frac{y'}{x} \Big]_{-\infty}^{\infty} = \frac{\rho_S}{2\epsilon_0}$$

If the point P were chosen on the negative x axis, then

$$E_x = -\frac{\rho_S}{2\epsilon_0}$$

for the field is always directed away from the positive charge. This difficulty in sign is usually overcome by specifying a unit vector $\mathbf{a}_N$, which is normal to the sheet and directed outward, or away from it. Then

$$\mathbf{E} = \frac{\rho_S}{2\epsilon_0} \mathbf{a}_N \tag{17}$$

This is a startling answer, for the field is constant in magnitude and direction. It is just as strong a million miles away from the sheet as it is right off the surface. Returning to our light analogy, we see that a uniform source of light on the ceiling of a very large room leads to just as much illumination on a square foot on the floor as it does on a square foot a few inches below the ceiling. If you desire greater illumination on this subject, it will do you no good to hold the book closer to such a light source.

If a second infinite sheet of charge, having a *negative* charge density $-\rho_S$, is located in the plane $x = a$, we may find the total field by adding the contribution of each sheet. In the region $x > a$,

$$\mathbf{E}_+ = \frac{\rho_S}{2\epsilon_0} \mathbf{a}_x \qquad \mathbf{E}_- = -\frac{\rho_S}{2\epsilon_0} \mathbf{a}_x \qquad \mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = 0$$

and for $x < 0$,

$$\mathbf{E}_+ = -\frac{\rho_S}{2\epsilon_0} \mathbf{a}_x \qquad \mathbf{E}_- = \frac{\rho_S}{2\epsilon_0} \mathbf{a}_x \qquad \mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = 0$$

and when $0 < x < a$,

$$\mathbf{E}_+ = \frac{\rho_S}{2\epsilon_0} \mathbf{a}_x \qquad \mathbf{E}_- = \frac{\rho_S}{2\epsilon_0} \mathbf{a}_x$$

and

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho_S}{\epsilon_0} \mathbf{a}_x \tag{18}$$

This is an important practical answer, for it is the field between the parallel plates of an air capacitor, provided the linear dimensions of the plates are very much greater than their separation and provided also that we are considering a point well removed from the edges. The field outside the capacitor, while not zero, as we found for the preceding ideal case, is usually negligible.

D2.6. Three infinite uniform sheets of charge are located in free space as follows: 3 nC/m^2 at $z = -4$, 6 nC/m^2 at $z = 1$, and -8 nC/m^2 at $z = 4$. Find **E** at the point: (*a*) $P_A(2, 5, -5)$; (*b*) $P_B(4, 2, -3)$; (*c*) $P_C(-1, -5, 2)$; (*d*) $P_D(-2, 4, 5)$.

Ans. $-56.5\mathbf{a}_z$; $283\mathbf{a}_z$; $961\mathbf{a}_z$; $56.5\mathbf{a}_z$ all V/m

2.6 STREAMLINES AND SKETCHES OF FIELDS

Illustrations

We now have vector equations for the electric field intensity resulting from several different charge configurations, and we have had little difficulty in interpreting the magnitude and direction of the field from the equations. Unfortunately, this simplicity cannot last much longer, for we have solved most of the simple cases and our new charge distributions must lead to more complicated expressions for the fields and more difficulty in visualizing the fields through the equations. However, it is true that one picture would be worth about a thousand words, if we just knew what picture to draw.

Consider the field about the line charge,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho$$

Figure 2.9*a* shows a cross-sectional view of the line charge and presents what might be our first effort at picturing the field—short line segments drawn here and there having lengths proportional to the magnitude of **E** and pointing in the direction of **E**. The figure fails to show the symmetry with respect to ϕ, so we try again in Figure 2.9*b* with a symmetrical location of the line segments. The real trouble now appears—the longest lines must be drawn in the most crowded region, and this also plagues us if we use line segments of equal length but of a thickness that is proportional to **E** (Figure 2.9*c*). Other schemes include drawing shorter lines to represent stronger fields (inherently misleading) and using intensity of color or different colors to represent stronger fields.

For the present, let us be content to show only the *direction* of **E** by drawing continuous lines, which are everywhere tangent to **E**, from the charge. Figure 2.9*d* shows this compromise. A symmetrical distribution of lines (one every 45°) indicates azimuthal symmetry, and arrowheads should be used to show direction.

These lines are usually called *streamlines*, although other terms such as flux lines and direction lines are also used. A small positive test charge placed at any point in this field and free to move would accelerate in the direction of the streamline passing through that point. If the field represented the velocity of a liquid or a gas (which, incidentally, would have to have a source at $\rho = 0$), small suspended particles in the liquid or gas would trace out the streamlines.

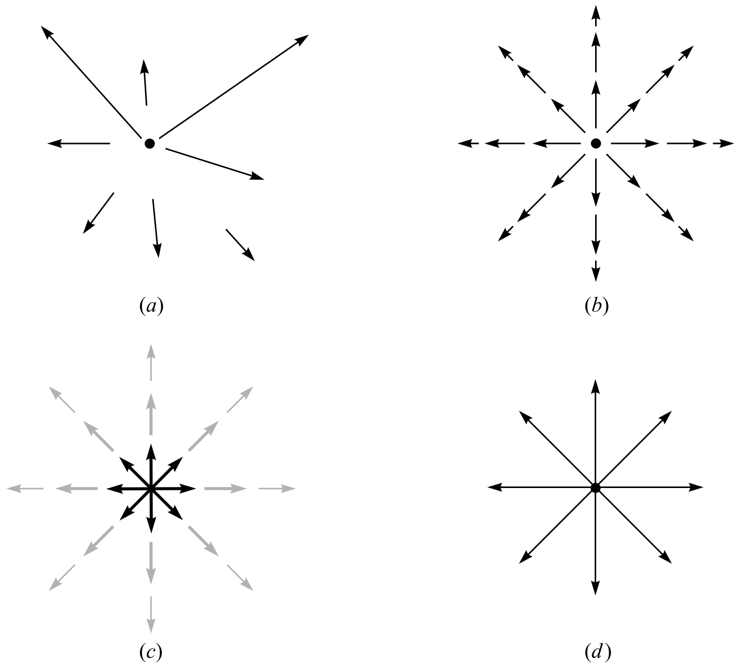

Figure 2.9 (*a*) One very poor sketch, (*b*) and (*c*) two fair sketches, and (*d*) the usual form of a streamline sketch. In the last form, the arrows show the direction of the field at every point along the line, and the spacing of the lines is inversely proportional to the strength of the field.

We will find out later that a bonus accompanies this streamline sketch, for the magnitude of the field can be shown to be inversely proportional to the spacing of the streamlines for some important special cases. The closer they are together, the stronger is the field. At that time we will also find an easier, more accurate method of making that type of streamline sketch.

If we attempted to sketch the field of the point charge, the variation of the field into and away from the page would cause essentially insurmountable difficulties; for this reason sketching is usually limited to two-dimensional fields.

In the case of the two-dimensional field, let us arbitrarily set $E_z = 0$. The streamlines are thus confined to planes for which z is constant, and the sketch is the same for any such plane. Several streamlines are shown in Figure 2.10, and the E_x and E_y components are indicated at a general point. It is apparent from the geometry that

$$\frac{E_y}{E_x} = \frac{dy}{dx} \tag{19}$$

A knowledge of the functional form of E_x and E_y (and the ability to solve the resultant differential equation) will enable us to obtain the equations of the streamlines.

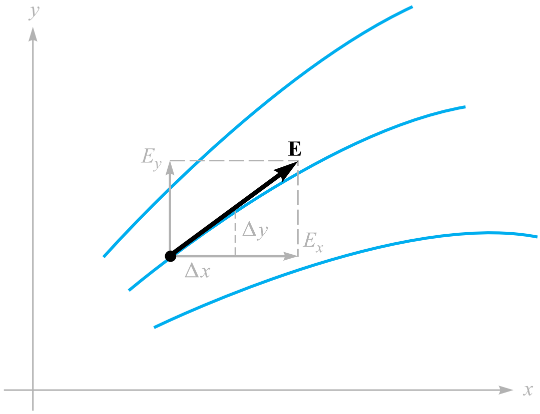

Figure 2.10 The equation of a streamline is obtained by solving the differential equation $E_y/E_x = dy/dx$.

As an illustration of this method, consider the field of the uniform line charge with $\rho_L = 2\pi\epsilon_0$,

$$\mathbf{E} = \frac{1}{\rho}\mathbf{a}_\rho$$

In rectangular coordinates,

$$\mathbf{E} = \frac{x}{x^2+y^2}\mathbf{a}_x + \frac{y}{x^2+y^2}\mathbf{a}_y$$

Thus we form the differential equation

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{y}{x} \qquad \text{or} \qquad \frac{dy}{y} = \frac{dx}{x}$$

Therefore,

$$\ln y = \ln x + C_1 \qquad \text{or} \qquad \ln y = \ln x + \ln C$$

from which the equations of the streamlines are obtained,

$$y = Cx$$

If we want to find the equation of one particular streamline, say one passing through $P(-2, 7, 10)$, we merely substitute the coordinates of that point into our equation and evaluate C. Here, $7 = C(-2)$, and $C = -3.5$, so $y = -3.5x$.

Each streamline is associated with a specific value of C, and the radial lines shown in Figure 2.9*d* are obtained when $C = 0, 1, -1$, and $1/C = 0$.

The equations of streamlines may also be obtained directly in cylindrical or spherical coordinates. A spherical coordinate example will be examined in Section 4.7.

D2.7. Find the equation of that streamline that passes through the point $P(1, 4, -2)$ in the field $\mathbf{E} = (a)\ \frac{-8x}{y}\mathbf{a}_x + \frac{4x^2}{y^2}\mathbf{a}_y$; $(b)\ 2e^{5x}[y(5x+1)\mathbf{a}_x + x\mathbf{a}_y]$.

Ans. $x^2 + 2y^2 = 33$; $y^2 = 15.7 + 0.4x - 0.08\ln(5x+1)$

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CHAPTER 2 PROBLEMS

WWW Quizzes

2.1 Three point charges are positioned in the x-y plane as follows: 5 nC at $y = 5$ cm, -10 nC at $y = -5$ cm, and 15 nC at $x = -5$ cm. Find the required x-y coordinates of a 20-nC fourth charge that will produce a zero electric field at the origin.

2.2 Point charges of 1 nC and -2 nC are located at (0, 0, 0) and (1, 1, 1), respectively, in free space. Determine the vector force acting on each charge.

2.3 Point charges of 50 nC each are located at $A(1, 0, 0)$, $B(-1, 0, 0)$, $C(0, 1, 0)$, and $D(0, -1, 0)$ in free space. Find the total force on the charge at A.

2.4 Eight identical point charges of Q C each are located at the corners of a cube of side length a, with one charge at the origin, and with the three nearest charges at $(a, 0, 0)$, $(0, a, 0)$, and $(0, 0, a)$. Find an expression for the total vector force on the charge at $P(a, a, a)$, assuming free space.

2.5 Let a point charge $Q_1 = 25$ nC be located at $P_1(4, -2, 7)$ and a charge $Q_2 = 60$ nC be at $P_2(-3, 4, -2)$. (a) If $\epsilon = \epsilon_0$, find $\mathbf{E}$ at $P_3(1, 2, 3)$. (b) At what point on the y axis is $E_x = 0$?

2.6 Two point charges of equal magnitude q are positioned at $z = \pm d/2$. (a) Find the electric field everywhere on the z axis; (b) find the electric field everywhere on the x axis; (c) repeat parts (a) and (b) if the charge at $z = -d/2$ is $-q$ instead of $+q$.

2.7 A 2-μC point charge is located at $A(4, 3, 5)$ in free space. Find E_ρ, E_ϕ, and E_z at $P(8, 12, 2)$.

2.8 A crude device for measuring charge consists of two small insulating spheres of radius a, one of which is fixed in position. The other is movable along the x axis and is subject to a restraining force kx, where k is a spring constant. The uncharged spheres are centered at $x = 0$ and $x = d$, the latter fixed. If the spheres are given equal and opposite charges of Q/C, obtain the expression by which Q may be found as a function of x. Determine the maximum charge that can be measured in terms of ϵ_0, k, and d, and state the separation of the spheres then. What happens if a larger charge is applied?

2.9 A 100-nC point charge is located at $A(-1, 1, 3)$ in free space. (a) Find the locus of all points $P(x, y, z)$ at which $E_x = 500$ V/m. (b) Find y_1 if $P(-2, y_1, 3)$ lies on that locus.

2.10 A charge of -1 nC is located at the origin in free space. What charge must be located at (2, 0, 0) to cause E_x to be zero at (3, 1, 1)?

2.11 A charge Q_0 located at the origin in free space produces a field for which $E_z = 1$ kV/m at point $P(-2, 1, -1)$. (a) Find Q_0. Find **E** at $M(1, 6, 5)$ in (b) rectangular coordinates; (c) cylindrical coordinates; (d) spherical coordinates.

2.12 Electrons are in random motion in a fixed region in space. During any 1 μs interval, the probability of finding an electron in a subregion of volume 10^{-15} m^2 is 0.27. What volume charge density, appropriate for such time durations, should be assigned to that subregion?

2.13 A uniform volume charge density of 0.2 μC/m^3 is present throughout the spherical shell extending from $r = 3$ cm to $r = 5$ cm. If $\rho_v = 0$ elsewhere, find (a) the total charge present throughout the shell, and (b) r_1 if half the total charge is located in the region 3 cm $< r < r_1$.

2.14 The electron beam in a certain cathode ray tube possesses cylindrical symmetry, and the charge density is represented by $\rho_v = -0.1/(\rho^2 + 10^{-8})$ pC/m^3 for $0 < \rho < 3 \times 10^{-4}$ m, and $\rho_v = 0$ for $\rho > 3 \times 10^{-4}$ m. (a) Find the total charge per meter along the length of the beam; (b) if the electron velocity is 5×10^7 m/s, and with one ampere defined as 1C/s, find the beam current.

2.15 A spherical volume having a 2-μm radius contains a uniform volume charge density of 10^{15} C/m^3. (a) What total charge is enclosed in the spherical volume? (b) Now assume that a large region contains one of these little spheres at every corner of a cubical grid 3 mm on a side and that there is no charge between the spheres. What is the average volume charge density throughout this large region?

2.16 Within a region of free space, charge density is given as $\rho_v = \frac{\rho_0 r \cos\theta}{a}$ C/m^3, where ρ_0 and a are constants. Find the total charge lying within (a) the sphere, $r \leq a$; (b) the cone, $r \leq a$, $0 \leq \theta \leq 0.1\pi$; ($c$) the region, $r \leq a$, $0 \leq \theta \leq 0.1\pi$, $0 \leq \phi \leq 0.2\pi$.

2.17 A uniform line charge of 16 nC/m is located along the line defined by $y = -2$, $z = 5$. If $\epsilon = \epsilon_0$: (*a*) find **E** at $P(1, 2, 3)$. (*b*) find **E** at that point in the $z = 0$ plane where the direction of **E** is given by $(1/3)\mathbf{a}_y - (2/3)\mathbf{a}_z$.

2.18 (a) Find **E** in the plane $z = 0$ that is produced by a uniform line charge, ρ_L, extending along the z axis over the range $-L < z < L$ in a cylindrical coordinate system. (b) If the finite line charge is approximated by an infinite line charge ($L \to \infty$), by what percentage is E_ρ in error if $\rho = 0.5L$? (*c*) Repeat (*b*) with $\rho = 0.1L$.

2.19 A uniform line charge of 2 μC/m is located on the z axis. Find **E** in rectangular coordinates at $P(1, 2, 3)$ if the charge exists from (*a*) $-\infty < z < \infty$; (*b*) $-4 \le z \le 4$.

2.20 A line charge of uniform charge density ρ_0 C/m and of length ℓ is oriented along the z axis at $-\ell/2 < z < \ell/2$. (*a*) Find the electric field strength, **E**, in magnitude and direction at any position along the x axis. (*b*) With the given line charge in position, find the force acting on an identical line charge that is oriented along the x axis at $\ell/2 < x < 3\ell/2$.

2.21 Two identical uniform line charges, with $\rho_l = 75$ nC/m, are located in free space at $x = 0$, $y = \pm 0.4$ m. What force per unit length does each line charge exert on the other?

2.22 Two identical uniform sheet charges with $\rho_s = 100$ nC/m^2 are located in free space at $z = \pm 2.0$ cm. What force per unit area does each sheet exert on the other?

2.23 Given the surface charge density, $\rho_s = 2\ \mu$C/m^2, existing in the region $\rho < 0.2$ m, $z = 0$, find **E** at (*a*) $P_A(\rho = 0, z = 0.5)$; (*b*) $P_B(\rho = 0, z = -0.5)$. Show that (*c*) the field along the z axis reduces to that of an infinite sheet charge at small values of z; (*d*) the z axis field reduces to that of a point charge at large values of z.

2.24 (*a*) Find the electric field on the z axis produced by an annular ring of uniform surface charge density ρ_s in free space. The ring occupies the region $z = 0$, $a \le \rho \le b$, $0 \le \phi \le 2\pi$ in cylindrical coordinates. (*b*) From your part (*a*) result, obtain the field of an infinite uniform sheet charge by taking appropriate limits.

2.25 Find **E** at the origin if the following charge distributions are present in free space: point charge, 12 nC, at $P(2, 0, 6)$; uniform line charge density, 3 nC/m, at $x = -2$, $y = 3$; uniform surface charge density, 0.2 nC/m^2 at $x = 2$.

2.26 A radially dependent surface charge is distributed on an infinite flat sheet in the x-y plane and is characterized in cylindrical coordinates by surface density $\rho_s = \rho_0/\rho$, where ρ_0 is a constant. Determine the electric field strength, **E**, everywhere on the z axis.

2.27 Given the electric field $\mathbf{E} = (4x - 2y)\mathbf{a}_x - (2x + 4y)\mathbf{a}_y$, find (*a*) the equation of the streamline that passes through the point $P(2, 3, -4)$; (*b*) a unit vector specifying the direction of $\mathbf{E}$ at $Q(3, -2, 5)$.

2.28 An electric dipole (discussed in detail in Section 4.7) consists of two point charges of equal and opposite magnitude $\pm Q$ spaced by distance d. With the charges along the z axis at positions $z = \pm d/2$ (with the positive charge at the positive z location), the electric field in spherical coordinates is given by $\mathbf{E}(r, \theta) = [Qd/(4\pi\epsilon_0 r^3)][2\cos\theta\mathbf{a}_r + \sin\theta\mathbf{a}_\theta]$, where $r >> d$. Using rectangular coordinates, determine expressions for the vector force on a point charge of magnitude q (*a*) at $(0, 0, z)$; (*b*) at $(0, y, 0)$.

2.29 If $\mathbf{E} = 20e^{-5y}(\cos 5x\mathbf{a}_x - \sin 5x\mathbf{a}_y)$, find (*a*) $|\mathbf{E}|$ at $P(\pi/6, 0.1, 2)$; (*b*) a unit vector in the direction of $\mathbf{E}$ at P; (*c*) the equation of the direction line passing through P.

2.30 For fields that do not vary with z in cylindrical coordinates, the equations of the streamlines are obtained by solving the differential equation $E_p/E_\phi = d\rho/(\rho d\phi)$. Find the equation of the line passing through the point $(2, 30^\circ, 0)$ for the field $\mathbf{E} = \rho\cos 2\phi\mathbf{a}_\rho - \rho\sin 2\phi\mathbf{a}_\phi$.

3

CHAPTER

Electric Flux Density, Gauss's Law, and Divergence

After drawing a few of the fields described in the previous chapter and becoming familiar with the concept of the streamlines that show the direction of the force on a test charge at every point, it is difficult to avoid giving these lines a physical significance and thinking of them as *flux* lines. No physical particle is projected radially outward from the point charge, and there are no steel tentacles reaching out to attract or repel an unwary test charge, but as soon as the streamlines are drawn on paper there seems to be a picture showing "something" is present.

It is very helpful to invent an *electric flux* that streams away symmetrically from a point charge and is coincident with the streamlines and to visualize this flux wherever an electric field is present.

This chapter introduces and uses the concept of electric flux and electric flux density to again solve several of the problems presented in Chapter 2. The work here turns out to be much easier, and this is due to the extremely symmetrical problems that we are solving. ■

3.1 ELECTRIC FLUX DENSITY

About 1837, the director of the Royal Society in London, Michael Faraday, became very interested in static electric fields and the effect of various insulating materials on these fields. This problem had been bothering him during the past ten years when he was experimenting in his now-famous work on induced electromotive force, which we will discuss in Chapter 10. With that subject completed, he had a pair of concentric metallic spheres constructed, the outer one consisting of two hemispheres that could be firmly clamped together. He also prepared shells of insulating material (or dielectric material, or simply *dielectric*) that would occupy the entire volume between the concentric spheres. We will immediately use his findings about dielectric materials, for we are restricting our attention to fields in free space until Chapter 6. At that time we will see that the materials he used will be classified as ideal dielectrics.

His experiment, then, consisted essentially of the following steps:

1. With the equipment dismantled, the inner sphere was given a known positive charge.
2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
3. The outer sphere was discharged by connecting it momentarily to ground.
4. The outer space was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.

Faraday found that the total charge on the outer sphere was equal in *magnitude* to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres. He concluded that there was some sort of "displacement" from the inner sphere to the outer which was independent of the medium, and we now refer to this flux as *displacement*, *displacement flux,* or simply *electric flux*.

Faraday's experiments also showed, of course, that a larger positive charge on the inner sphere induced a correspondingly larger negative charge on the outer sphere, leading to a direct proportionality between the electric flux and the charge on the inner sphere. The constant of proportionality is dependent on the system of units involved, and we are fortunate in our use of SI units, because the constant is unity. If electric flux is denoted by Ψ (psi) and the total charge on the inner sphere by Q, then for Faraday's experiment

$$\boxed{\Psi = Q}$$

and the electric flux Ψ is measured in coulombs.

We can obtain more quantitative information by considering an inner sphere of radius a and an outer sphere of radius b, with charges of Q and $-Q$, respectively (Figure 3.1). The paths of electric flux Ψ extending from the inner sphere to the outer sphere are indicated by the symmetrically distributed streamlines drawn radially from one sphere to the other.

At the surface of the inner sphere, Ψ coulombs of electric flux are produced by the charge $Q(=\Psi)$ Cs distributed uniformly over a surface having an area of $4\pi a^2$ m^2. The density of the flux at this surface is $\Psi/4\pi a^2$ or $Q/4\pi a^2$ C/m^2, and this is an important new quantity.

Electric flux density, measured in coulombs per square meter (sometimes described as "lines per square meter," for each line is due to one coulomb), is given the letter **D**, which was originally chosen because of the alternate names of *displacement flux density* or *displacement density*. Electric flux density is more descriptive, however, and we will use the term consistently.

The electric flux density **D** is a vector field and is a member of the "flux density" class of vector fields, as opposed to the "force fields" class, which includes the electric

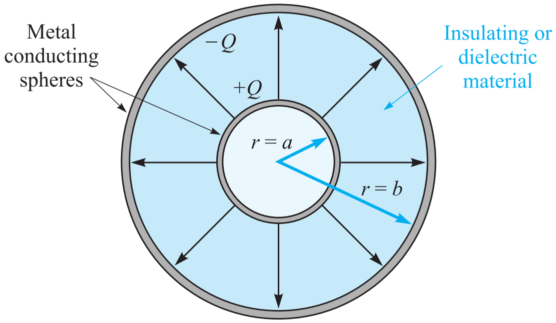

Figure 3.1 The electric flux in the region between a pair of charged concentric spheres. The direction and magnitude of **D** are not functions of the dielectric between the spheres.

field intensity **E**. The direction of **D** at a point is the direction of the flux lines at that point, and the magnitude is given by the number of flux lines crossing a surface normal to the lines divided by the surface area.

Referring again to Figure 3.1, the electric flux density is in the radial direction and has a value of

$$\mathbf{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\mathbf{a}_r \qquad \text{(inner sphere)}$$

$$\mathbf{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\mathbf{a}_r \qquad \text{(outer sphere)}$$

and at a radial distance r, where $a \le r \le b$,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

If we now let the inner sphere become smaller and smaller, while still retaining a charge of Q, it becomes a point charge in the limit, but the electric flux density at a point r meters from the point charge is still given by

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r \tag{1}$$

for Q lines of flux are symmetrically directed outward from the point and pass through an imaginary spherical surface of area $4\pi r^2$.

This result should be compared with Section 2.2, Eq. (9), the radial electric field intensity of a point charge in free space,

$$\mathbf{E} = \frac{Q}{4\pi \epsilon_0 r^2}\mathbf{a}_r$$

In free space, therefore,

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E}} \quad \text{(free space only)} \tag{2}$$

Although (2) is applicable only to a vacuum, it is not restricted solely to the field of a point charge. For a general volume charge distribution in free space,

$$\boxed{\mathbf{E} = \int_{\text{vol}} \frac{\rho_v dv}{4\pi\epsilon_0 R^2}\mathbf{a}_R} \quad \text{(free space only)} \tag{3}$$

where this relationship was developed from the field of a single point charge. In a similar manner, (1) leads to

$$\boxed{\mathbf{D} = \int_{\text{vol}} \frac{\rho_v dv}{4\pi R^2}\mathbf{a}_R} \tag{4}$$

and (2) is therefore true for any free-space charge configuration; we will consider (2) as defining **D** in free space.

As a preparation for the study of dielectrics later, it might be well to point out now that, for a point charge embedded in an infinite ideal dielectric medium, Faraday's results show that (1) is still applicable, and thus so is (4). Equation (3) is not applicable, however, and so the relationship between **D** and **E** will be slightly more complicated than (2).

Because **D** is directly proportional to **E** in free space, it does not seem that it should really be necessary to introduce a new symbol. We do so for a few reasons. First, **D** is associated with the flux concept, which is an important new idea. Second, the **D** fields we obtain will be a little simpler than the corresponding **E** fields, because ϵ_0 does not appear.

D3.1. Given a 60- C point charge located at the origin, find the total electric flux passing through: (*a*) that portion of the sphere $r = 26$ cm bounded by $0 < \theta < \frac{\pi}{2}$ and $0 < \phi < \frac{\pi}{2}$; (*b*) the closed surface defined by $\rho = 26$ cm and $z = \pm 26$ cm; (*c*) the plane $z = 26$ cm.

Ans. 7.5 C; 60 C; 30 C

D3.2. Calculate **D** in rectangular coordinates at point $P(2, -3, 6)$ produced by: (*a*) a point charge $Q_A = 55$ mC at $Q(-2, 3, -6)$; (*b*) a uniform line charge $\rho_{LB} = 20$ mC/m on the x axis; (*c*) a uniform surface charge density $\rho_{SC} = 120$ C/m^2 on the plane $z = -5$ m.

Ans. $6.38\mathbf{a}_x - 9.57\mathbf{a}_y + 19.14\mathbf{a}_z$ C/m^2; $-212\mathbf{a}_y + 424\mathbf{a}_z$ C/m^2; $60\mathbf{a}_z$ C/m^2

3.2 GAUSS'S LAW

The results of Faraday's experiments with the concentric spheres could be summed up as an experimental law by stating that the electric flux passing through any imaginary spherical surface lying between the two conducting spheres is equal to the charge enclosed within that imaginary surface. This enclosed charge is distributed on the surface of the inner sphere, or it might be concentrated as a point charge at the center of the imaginary sphere. However, because one coulomb of electric flux is produced by one coulomb of charge, the inner conductor might just as well have been a cube or a brass door key and the total induced charge on the outer sphere would still be the same. Certainly the flux density would change from its previous symmetrical distribution to some unknown configuration, but $+Q$ coulombs on any inner conductor would produce an induced charge of $-Q$ coulombs on the surrounding sphere. Going one step further, we could now replace the two outer hemispheres by an empty (but completely closed) soup can. Q coulombs on the brass door key would produce $\Psi = Q$ lines of electric flux and would induce $-Q$ coulombs on the tin can.[1]

These generalizations of Faraday's experiment lead to the following statement, which is known as *Gauss's law*:

> *The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.*

The contribution of Gauss, one of the greatest mathematicians the world has ever produced, was actually not in stating the law as we have, but in providing a mathematical form for this statement, which we will now obtain.

Let us imagine a distribution of charge, shown as a cloud of point charges in Figure 3.2, surrounded by a closed surface of any shape. The closed surface may be the surface of some real material, but more generally it is any closed surface we wish to visualize. If the total charge is Q, then Q coulombs of electric flux will pass through the enclosing surface. At every point on the surface the electric-flux-density vector $\mathbf{D}$ will have some value $\mathbf{D}_S$, where the subscript S merely reminds us that $\mathbf{D}$ must be evaluated at the surface, and $\mathbf{D}_S$ will in general vary in magnitude and direction from one point on the surface to another.

We must now consider the nature of an incremental element of the surface. An incremental element of area ΔS is very nearly a portion of a plane surface, and the complete description of this surface element requires not only a statement of its magnitude ΔS but also of its orientation in space. In other words, the incremental surface element is a vector quantity. The only unique direction that may be associated with $\Delta\mathbf{S}$ is the direction of the normal to that plane which is tangent to the surface at the point in question. There are, of course, two such normals, and the ambiguity is removed by specifying the outward normal whenever the surface is closed and "outward" has a specific meaning.

[1] If it were a perfect insulator, the soup could even be left in the can without any difference in the results.

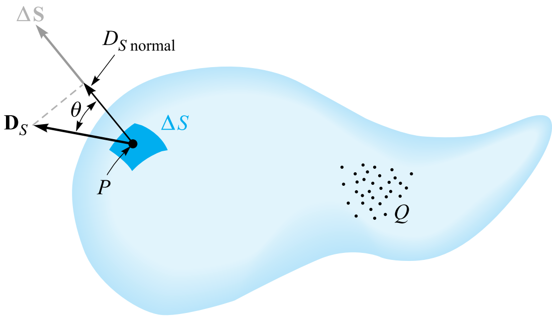

Figure 3.2 The electric flux density $\mathbf{D}_S$ at P arising from charge Q. The total flux passing through ΔS is $\mathbf{D}_S \cdot \Delta\mathbf{S}$.

At any point P, consider an incremental element of surface ΔS and let $\mathbf{D}_S$ make an angle θ with $\Delta\mathbf{S}$, as shown in Figure 3.2. The flux crossing ΔS is then the product of the normal component of $\mathbf{D}_S$ and $\Delta\mathbf{S}$,

$$\Delta\Psi = \text{flux crossing } \Delta S = D_{S,\text{norm}} \Delta S = D_S \cos\theta \, \Delta S = \mathbf{D}_S \cdot \Delta\mathbf{S}$$

where we are able to apply the definition of the dot product developed in Chapter 1.

The *total* flux passing through the closed surface is obtained by adding the differential contributions crossing each surface element $\Delta\mathbf{S}$,

$$\Psi = \int d\Psi = \oint_{\substack{\text{closed}\\ \text{surface}}} \mathbf{D}_S \cdot d\mathbf{S}$$

The resultant integral is a *closed surface integral*, and since the surface element $d\mathbf{S}$ always involves the differentials of two coordinates, such as $dx\,dy$, $\rho\,d\phi\,d\rho$, or $r^2 \sin\theta\,d\theta\,d\phi$, the integral is a double integral. Usually only one integral sign is used for brevity, and we will always place an S below the integral sign to indicate a surface integral, although this is not actually necessary, as the differential $d\mathbf{S}$ is automatically the signal for a surface integral. One last convention is to place a small circle on the integral sign itself to indicate that the integration is to be performed over a *closed* surface. Such a surface is often called a *gaussian surface*. We then have the mathematical formulation of Gauss's law,

$$\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q \qquad (5)$$

The charge enclosed might be several point charges, in which case

$$Q = \Sigma Q n$$

or a line charge,

$$Q = \int \rho_L \, dL$$

or a surface charge,

$$Q = \int_S \rho_S dS \qquad \text{(not necessarily a closed surface)}$$

or a volume charge distribution,

$$Q = \int_{\text{vol}} \rho_v \, dv$$

The last form is usually used, and we should agree now that it represents any or all of the other forms. With this understanding, Gauss's law may be written in terms of the charge distribution as

$$\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v \, dv \tag{6}$$

a mathematical statement meaning simply that the total electric flux through any closed surface is equal to the charge enclosed.

EXAMPLE 3.1

To illustrate the application of Gauss's law, let us check the results of Faraday's experiment by placing a point charge Q at the origin of a spherical coordinate system (Figure 3.3) and by choosing our closed surface as a sphere of radius a.

Solution. We have, as before,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

At the surface of the sphere,

$$\mathbf{D}_S = \frac{Q}{4\pi a^2}\mathbf{a}_r$$

The differential element of area on a spherical surface is, in spherical coordinates from Chapter 1,

$$dS = r^2 \sin\theta \, d\theta \, d\phi = a^2 \sin\theta \, d\theta \, d\phi$$

or

$$d\mathbf{S} = a^2 \sin\theta \, d\theta \, d\phi \, \mathbf{a}_r$$

The integrand is

$$\mathbf{D}_S \cdot d\mathbf{S} = \frac{Q}{4\pi a^2} a^2 \sin\theta \, d\theta \, d\phi \mathbf{a}_r \cdot \mathbf{a}_r = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

leading to the closed surface integral

$$\int_{\phi=0}^{\phi=2\pi} \int_{\theta=\phi}^{\theta=\pi} \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

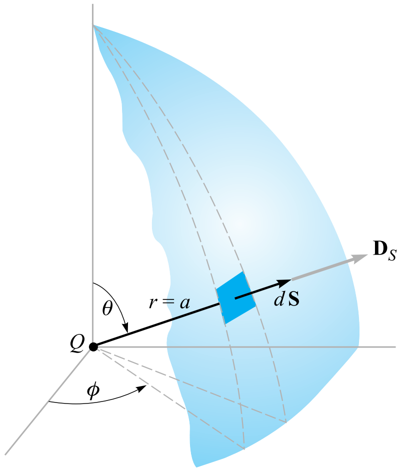

Figure 3.3 Applying Gauss's law to the field of a point charge Q on a spherical closed surface of radius a. The electric flux density **D** is everywhere normal to the spherical surface and has a constant magnitude at every point on it.

where the limits on the integrals have been chosen so that the integration is carried over the entire surface of the sphere once.[2] Integrating gives

$$\int_0^{2\pi} \frac{Q}{4\pi}\left(-\cos\theta\right)_0^{\pi} d\phi = \int_0^{2\pi} \frac{Q}{2\pi} d\phi = Q$$

Animations

and we obtain a result showing that Q coulombs of electric flux are crossing the surface, as we should since the enclosed charge is Q coulombs.

D3.3. Given the electric flux density, $\mathbf{D} = 0.3r^2\mathbf{a}_r$ nC/m^2 in free space: (a) find **E** at point $P(r = 2, \theta = 25^\circ, \phi = 90^\circ)$; ($b$) find the total charge within the sphere $r = 3$; (c) find the total electric flux leaving the sphere $r = 4$.

Ans. $135.5\mathbf{a}_r$ V/m; 305 nC; 965 nC

D3.4. Calculate the total electric flux leaving the cubical surface formed by the six planes $x, y, z = \pm 5$ if the charge distribution is: (a) two point charges, 0.1 C at $(1, -2, 3)$ and $\frac{1}{7}$ C at $(-1, 2, -2)$; (b) a uniform line charge of π C/m at $x = -2$, $y = 3$; (c) a uniform surface charge of 0.1 C/m^2 on the plane $y = 3x$.

Ans. 0.243 C; 31.4 C; 10.54 C

[2] Note that if θ and ϕ both cover the range from 0 to 2π, the spherical surface is covered twice.

3.3 APPLICATION OF GAUSS'S LAW: SOME SYMMETRICAL CHARGE DISTRIBUTIONS

We now consider how we may use Gauss's law,

$$Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}$$

to determine $\mathbf{D}_S$ if the charge distribution is known. This is an example of an integral equation in which the unknown quantity to be determined appears inside the integral.

The solution is easy if we are able to choose a closed surface which satisfies two conditions:

1. $\mathbf{D}_S$ is everywhere either normal or tangential to the closed surface, so that $\mathbf{D}_S \cdot d\mathbf{S}$ becomes either $D_S dS$ or zero, respectively.
2. On that portion of the closed surface for which $\mathbf{D}_S \cdot d\mathbf{S}$ is not zero, $D_S =$ constant.

This allows us to replace the dot product with the product of the scalars D_S and dS and then to bring D_S outside the integral sign. The remaining integral is then $\int_S dS$ over that portion of the closed surface which $\mathbf{D}_S$ crosses normally, and this is simply the area of this section of that surface. Only a knowledge of the symmetry of the problem enables us to choose such a closed surface.

Let us again consider a point charge Q at the origin of a spherical coordinate system and decide on a suitable closed surface which will meet the two requirements previously listed. The surface in question is obviously a spherical surface, centered at the origin and of any radius r. $\mathbf{D}_S$ is everywhere normal to the surface; D_S has the same value at all points on the surface.

Then we have, in order,

$$Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \oint_{\text{sph}} D_S dS$$

$$= D_S \oint_{\text{sph}} dS = D_S \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= 4\pi r^2 D_S$$

and hence

$$D_S = \frac{Q}{4\pi r^2}$$

Because r may have any value and because $\mathbf{D}_S$ is directed radially outward,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r \qquad \mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

which agrees with the results of Chapter 2. The example is a trivial one, and the objection could be raised that we had to know that the field was symmetrical and directed radially outward before we could obtain an answer. This is true, and that leaves the inverse-square-law relationship as the only check obtained from Gauss's law. The example does, however, serve to illustrate a method which we may apply to other problems, including several to which Coulomb's law is almost incapable of supplying an answer.

Are there any other surfaces which would have satisfied our two conditions? The student should determine that such simple surfaces as a cube or a cylinder do not meet the requirements.

As a second example, let us reconsider the uniform line charge distribution ρ_L lying along the z axis and extending from $-\infty$ to $+\infty$. We must first know the symmetry of the field, and we may consider this knowledge complete when the answers to these two questions are known:

1. With which coodinates does the field vary (or of what variables is D a function)?
2. Which components of **D** are present?

In using Gauss's law, it is not a question of using symmetry to simplify the solution, for the application of Gauss's law depends on symmetry, and *if we cannot show that symmetry exists then we cannot use Gauss's law* to obtain a solution. The preceding two questions now become "musts."

From our previous discussion of the uniform line charge, it is evident that only the radial component of **D** is present, or

$$\mathbf{D} = D_\rho \mathbf{a}_\rho$$

and this component is a function of ρ only.

$$D_\rho = f(\rho)$$

The choice of a closed surface is now simple, for a cylindrical surface is the only surface to which D_ρ is everywhere normal, and it may be closed by plane surfaces normal to the z axis. A closed right circular cylinder of radius ρ extending from $z = 0$ to $z = L$ is shown in Figure 3.4.

We apply Gauss's law,

$$Q = \oint_{\text{cyl}} \mathbf{D}_S \cdot d\mathbf{S} = D_S \int_{\text{sides}} dS + 0 \int_{\text{top}} dS + 0 \int_{\text{bottom}} dS$$

$$= D_S \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = D_S 2\pi \rho L$$

and obtain

$$D_S = D_\rho = \frac{Q}{2\pi \rho L}$$

In terms of the charge density ρ_L, the total charge enclosed is

$$Q = \rho_L L$$

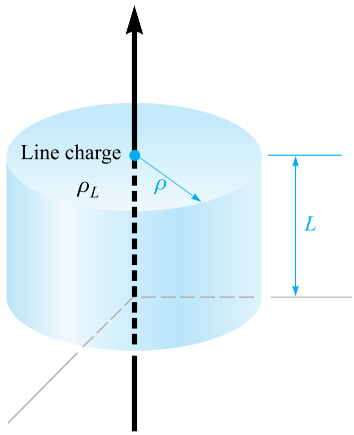

Figure 3.4 The gaussian surface for an infinite uniform line charge is a right circular cylinder of length L and radius ρ. **D** is constant in magnitude and everywhere perpendicular to the cylindrical surface; **D** is parallel to the end faces.

giving

$$D_\rho = \frac{\rho_L}{2\pi\rho}$$

or

$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0\rho}$$

Comparing with Section 2.4, Eq. (16), shows that the correct result has been obtained and with much less work. Once the appropriate surface has been chosen, the integration usually amounts only to writing down the area of the surface at which **D** is normal.

The problem of a coaxial cable is almost identical with that of the line charge and is an example that is extremely difficult to solve from the standpoint of Coulomb's law. Suppose that we have two coaxial cylindrical conductors, the inner of radius a and the outer of radius b, each infinite in extent (Figure 3.5). We will assume a charge distribution of ρ_S on the outer surface of the inner conductor.

Symmetry considerations show us that only the D_ρ component is present and that it can be a function only of ρ. A right circular cylinder of length L and radius ρ, where $a < \rho < b$, is necessarily chosen as the gaussian surface, and we quickly have

$$Q = D_S 2\pi\rho L$$

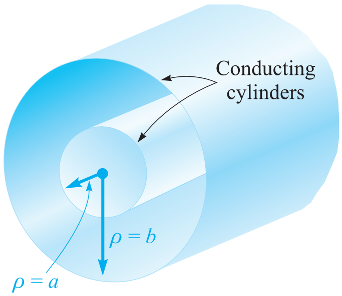

Figure 3.5 The two coaxial cylindrical conductors forming a coaxial cable provide an electric flux density within the cylinders, given by $D_\rho = a\rho_S/\rho$.

The total charge on a length L of the inner conductor is

$$Q = \int_{z=0}^{L}\int_{\phi=0}^{2\pi} \rho_S a \, d\phi \, dz = 2\pi a L \rho_S$$

from which we have

$$D_S = \frac{a\rho_S}{\rho} \qquad \mathbf{D} = \frac{a\rho_S}{\rho}\mathbf{a}_\rho \qquad (a < \rho < b)$$

This result might be expressed in terms of charge per unit length because the inner conductor has $2\pi a \rho_S$ coulombs on a meter length, and hence, letting $\rho_L = 2\pi a \rho_S$,

$$\mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho$$

Illustrations

and the solution has a form identical with that of the infinite line charge.

Because every line of electric flux starting from the charge on the inner cylinder must terminate on a negative charge on the inner surface of the outer cylinder, the total charge on that surface must be

$$Q_{\text{outer cyl}} = -2\pi a L \rho_{S,\text{inner cyl}}$$

and the surface charge on the outer cylinder is found as

$$2\pi b L \rho_{S,\text{outer cyl}} = -2\pi a L \rho_{S,\text{inner cyl}}$$

or

$$\rho_{S,\text{outer cyl}} = -\frac{a}{b}\rho_{S,\text{inner cyl}}$$

What would happen if we should use a cylinder of radius ρ, $\rho > b$, for the gaussian surface? The total charge enclosed would then be zero, for there are equal and opposite charges on each conducting cylinder. Hence

$$0 = D_S 2\pi\rho L \qquad (\rho > b)$$

$$D_S = 0 \qquad (\rho > b)$$

An identical result would be obtained for $\rho < a$. Thus the coaxial cable or capacitor has no external field (we have proved that the outer conductor is a "shield"), and there is no field within the center conductor.

Our result is also useful for a *finite* length of coaxial cable, open at both ends, provided the length L is many times greater than the radius b so that the nonsymmetrical conditions at the two ends do not appreciably affect the solution. Such a device is also termed a *coaxial capacitor*. Both the coaxial cable and the coaxial capacitor will appear frequently in the work that follows.

EXAMPLE 3.2

Let us select a 50-cm length of coaxial cable having an inner radius of 1 mm and an outer radius of 4 mm. The space between conductors is assumed to be filled with air. The total charge on the inner conductor is 30 nC. We wish to know the charge density on each conductor, and the **E** and **D** fields.

Solution. We begin by finding the surface charge density on the inner cylinder,

$$\rho_{S,\text{inner cyl}} = \frac{Q_{\text{inner cyl}}}{2\pi a L} = \frac{30 \times 10^{-9}}{2\pi(10^{-3})(0.5)} = 9.55\ \mu\text{C/m}^2$$

The negative charge density on the inner surface of the outer cylinder is

$$\rho_{S,\text{outer cyl}} = \frac{Q_{\text{outer cyl}}}{2\pi b L} = \frac{-30 \times 10^{-9}}{2\pi(4 \times 10^{-3})(0.5)} = -2.39\ \mu\text{C/m}^2$$

The internal fields may therefore be calculated easily:

$$D_\rho = \frac{a\rho_S}{\rho} = \frac{10^{-3}(9.55 \times 10^{-6})}{\rho} = \frac{9.55}{\rho}\ \text{nC/m}^2$$

and

$$E_\rho = \frac{D_\rho}{\epsilon_0} = \frac{9.55 \times 10^{-9}}{8.854 \times 10^{-12}\rho} = \frac{1079}{\rho}\ \text{V/m}$$

Both of these expressions apply to the region where $1 < \rho < 4$ mm. For $\rho < 1$ mm or $\rho > 4$ mm, **E** and **D** are zero.

D3.5. A point charge of 0.25 C is located at $r = 0$, and uniform surface charge densities are located as follows: 2 mC/m^2 at $r = 1$ cm, and -0.6 mC/m^2 at $r = 1.8$ cm. Calculate **D** at: (a) $r = 0.5$ cm; (b) $r = 1.5$ cm; (c) $r = 2.5$ cm. (d) What uniform surface charge density should be established at $r = 3$ cm to cause $\mathbf{D} = 0$ at $r = 3.5$ cm?

Ans. $796\mathbf{a}_r$ C/m^2; $977\mathbf{a}_r$ C/m^2; $40.8\mathbf{a}_r$ C/m^2; -28.3 C/m^2

3.4 APPLICATION OF GAUSS'S LAW: DIFFERENTIAL VOLUME ELEMENT

We are now going to apply the methods of Gauss's law to a slightly different type of problem—one that does not possess any symmetry at all. At first glance, it might seem that our case is hopeless, for without symmetry, a simple gaussian surface cannot be chosen such that the normal component of **D** is constant or zero everywhere on the surface. Without such a surface, the integral cannot be evaluated. There is only one way to circumvent these difficulties and that is to choose such a very small closed surface that **D** is *almost* constant over the surface, and the small change in **D** may be adequately represented by using the first two terms of the Taylor's-series expansion for **D**. The result will become more nearly correct as the volume enclosed by the gaussian surface decreases, and we intend eventually to allow this volume to approach zero.

This example also differs from the preceding ones in that we will not obtain the value of **D** as our answer but will instead receive some extremely valuable information about the way **D** varies in the region of our small surface. This leads directly to one of Maxwell's four equations, which are basic to all electromagnetic theory.

Let us consider any point P, shown in Figure 3.6, located by a rectangular coordinate system. The value of **D** at the point P may be expressed in rectangular components, $\mathbf{D}_0 = D_{x0}\mathbf{a}_x + D_{y0}\mathbf{a}_y + D_{z0}\mathbf{a}_z$. We choose as our closed surface the small rectangular box, centered at P, having sides of lengths Δx, Δy, and Δz, and apply Gauss's law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

In order to evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

Consider the first of these in detail. Because the surface element is very small, **D** is essentially constant (over *this* portion of the entire closed surface) and

$$\begin{aligned} \int_{\text{front}} &\doteq \mathbf{D}_{\text{front}} \cdot \Delta\mathbf{S}_{\text{front}} \\ &\doteq \mathbf{D}_{\text{front}} \cdot \Delta y\, \Delta z\, \mathbf{a}_x \\ &\doteq D_{\text{x,front}} \Delta y\, \Delta z \end{aligned}$$

where we have only to approximate the value of D_x at this front face. The front face is at a distance of $\Delta x/2$ from P, and hence

$$\begin{aligned} D_{x,\text{front}} &\doteq D_{x0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{ with } x \\ &\doteq D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \end{aligned}$$

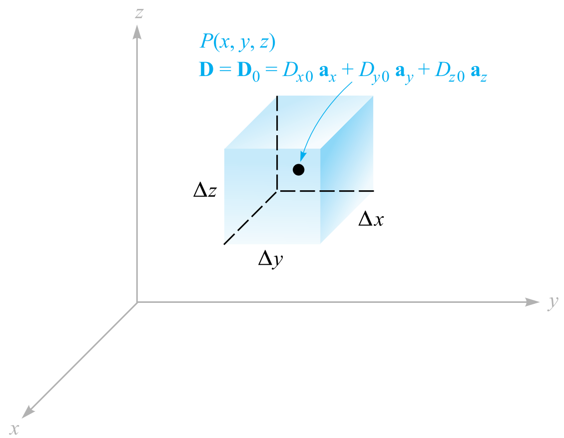

Figure 3.6 A differential-sized gaussian surface about the point P is used to investigate the space rate of change of **D** in the neighborhood of P.

where D_{x0} is the value of D_x at P, and where a partial derivative must be used to express the rate of change of D_x with x, as D_x in general also varies with y and z. This expression could have been obtained more formally by using the constant term and the term involving the first derivative in the Taylor's-series expansion for D_x in the neighborhood of P.

We now have

$$\int_{\text{front}} \doteq \left(D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \, \Delta z$$

Consider now the integral over the back surface,

$$\begin{aligned} \int_{\text{back}} &\doteq \mathbf{D}_{\text{back}} \cdot \Delta \mathbf{S}_{\text{back}} \\ &\doteq \mathbf{D}_{\text{back}} \cdot (-\Delta y \, \Delta z \, \mathbf{a}_x) \\ &\doteq -D_{x,\text{back}} \Delta y \, \Delta z \end{aligned}$$

and

$$D_{x,\text{back}} \doteq D_{x0} - \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

giving

$$\int_{\text{back}} \doteq \left(-D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \, \Delta z$$

If we combine these two integrals, we have

$$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x} \Delta x \, \Delta y \, \Delta z$$

By exactly the same process we find that

$$\int_{\text{right}} + \int_{\text{left}} \doteq \frac{\partial D_y}{\partial y} \Delta x\, \Delta y\, \Delta z$$

and

$$\int_{\text{top}} + \int_{\text{bottom}} \doteq \frac{\partial D_z}{\partial z} \Delta x\, \Delta y\, \Delta z$$

and these results may be collected to yield

$$\oint_S \mathbf{D} \cdot d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x\, \Delta y\, \Delta z$$

or

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v \tag{7}$$

The expression is an approximation which becomes better as Δv becomes smaller, and in the following section we shall let the volume Δv approach zero. For the moment, we have applied Gauss's law to the closed surface surrounding the volume element Δv and have as a result the approximation (7) stating that

$$\text{Charge enclosed in volume } \Delta v \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \times \text{volume } \Delta v \tag{8}$$

EXAMPLE 3.3

Find an approximate value for the total charge enclosed in an incremental volume of 10^{-9} m^3 located at the origin, if $\mathbf{D} = e^{-x} \sin y\, \mathbf{a}_x - e^{-x} \cos y\, \mathbf{a}_y + 2z\mathbf{a}_z$ C/m^2.

Solution. We first evaluate the three partial derivatives in (8):

$$\frac{\partial D_x}{\partial x} = -e^{-x} \sin y$$

$$\frac{\partial D_y}{\partial y} = e^{-x} \sin y$$

$$\frac{\partial D_z}{\partial z} = 2$$

At the origin, the first two expressions are zero, and the last is 2. Thus, we find that the charge enclosed in a small volume element there must be approximately $2\Delta v$. If Δv is 10^{-9} m^3, then we have enclosed about 2 nC.

D3.6. In free space, let $\mathbf{D} = 8xyz^4\mathbf{a}_x + 4x^2z^4\mathbf{a}_y + 16x^2yz^3\mathbf{a}_z$ pC/m^2. (*a*) Find the total electric flux passing through the rectangular surface $z = 2$, $0 < x < 2$, $1 < y < 3$, in the $\mathbf{a}_z$ direction. (*b*) Find $\mathbf{E}$ at $P(2, -1, 3)$. (*c*) Find an approximate value for the total charge contained in an incremental sphere located at $P(2, -1, 3)$ and having a volume of 10^{-12} m^3.

Ans. 1365 pC; $-146.4\mathbf{a}_x + 146.4\mathbf{a}_y - 195.2\mathbf{a}_z$ V/m; -2.38×10^{-21} C

3.5 DIVERGENCE AND MAXWELL'S FIRST EQUATION

We will now obtain an exact relationship from (7), by allowing the volume element $\Delta\nu$ to shrink to zero. We write this equation as

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta\nu \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta\nu} = \lim_{\Delta\nu \to 0} \frac{Q}{\Delta\nu} = \rho_\nu \tag{9}$$

in which the charge density, ρ_ν, is identified in the second equality.

The methods of the previous section could have been used on any vector $\mathbf{A}$ to find $\oint_S \mathbf{A} \cdot d\mathbf{S}$ for a small closed surface, leading to

$$\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) = \lim_{\Delta\nu \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta\nu} \tag{10}$$

where $\mathbf{A}$ could represent velocity, temperature gradient, force, or any other vector field.

This operation appeared so many times in physical investigations in the last century that it received a descriptive name, *divergence*. The divergence of $\mathbf{A}$ is defined as

$$\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \lim_{\Delta\nu \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta\nu} \tag{11}$$

and is usually abbreviated div $\mathbf{A}$. The physical interpretation of the divergence of a vector is obtained by describing carefully the operations implied by the right-hand side of (11), where we shall consider $\mathbf{A}$ to be a member of the flux-density family of vectors in order to aid the physical interpretation.

The divergence of the vector flux density $\mathbf{A}$ *is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.*

The physical interpretation of divergence afforded by this statement is often useful in obtaining qualitative information about the divergence of a vector field without resorting to a mathematical investigation. For instance, let us consider the divergence of the velocity of water in a bathtub after the drain has been opened. The net outflow of water through *any* closed surface lying entirely within the water must be zero, for water is essentially incompressible, and the water entering and leaving different regions of the closed surface must be equal. Hence the divergence of this velocity is zero.

If, however, we consider the velocity of the air in a tire that has just been punctured by a nail, we realize that the air is expanding as the pressure drops, and that consequently there is a net outflow from any closed surface lying within the tire. The divergence of this velocity is therefore greater than zero.

A positive divergence for any vector quantity indicates a *source* of that vector quantity at that point. Similarly, a negative divergence indicates a *sink*. Because the divergence of the water velocity above is zero, no source or sink exists.[3] The expanding air, however, produces a positive divergence of the velocity, and each interior point may be considered a source.

Writing (9) with our new term, we have

$$\text{div}\,\mathbf{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \qquad \text{(rectangular)} \tag{12}$$

This expression is again of a form that does not involve the charge density. It is the result of applying the definition of divergence (11) to a differential volume element in *rectangular coordinates*.

If a differential volume unit $\rho\, d\rho\, d\phi\, dz$ in cylindrical coordinates, or $r^2 \sin\theta\, dr\, d\theta\, d\phi$ in spherical coordinates, had been chosen, expressions for divergence involving the components of the vector in the particular coordinate system and involving partial derivatives with respect to the variables of that system would have been obtained. These expressions are obtained in Appendix A and are given here for convenience:

$$\text{div}\,\mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \qquad \text{(cylindrical)} \tag{13}$$

$$\text{div}\,\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi} \qquad \text{(spherical)} \tag{14}$$

These relationships are also shown inside the back cover for easy reference.

It should be noted that the divergence is an operation which is performed on a vector, but that the result is a scalar. We should recall that, in a somewhat similar way, the dot or scalar product was a multiplication of two vectors which yielded a scalar.

For some reason, it is a common mistake on meeting divergence for the first time to impart a vector quality to the operation by scattering unit vectors around in

[3] Having chosen a differential element of volume within the water, the gradual decrease in water level with time will eventually cause the volume element to lie above the surface of the water. At the instant the surface of the water intersects the volume element, the divergence is positive and the small volume is a source. This complication is avoided above by specifying an integral point.

the partial derivatives. Divergence merely tells us *how much* flux is leaving a small volume on a per-unit-volume basis; no direction is associated with it.

We can illustrate the concept of divergence by continuing with the example at the end of Section 3.4.

EXAMPLE 3.4

Find div **D** at the origin if $\mathbf{D} = e^{-x}\sin y\,\mathbf{a}_x - e^{-x}\cos y\,\mathbf{a}_y + 2z\mathbf{a}_z$.

Solution. We use (10) to obtain

$$\begin{aligned}\text{div}\,\mathbf{D} &= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\\ &= -e^{-x}\sin\,y + e^{-x}\sin\,y + 2 = 2\end{aligned}$$

The value is the constant 2, regardless of location.

If the units of **D** are C/m^2, then the units of div **D** are C/m^3. This is a volume charge density, a concept discussed in the next section.

D3.7. In each of the following parts, find a numerical value for div **D** at the point specified: (*a*) $\mathbf{D} = (2xyz - y^2)\mathbf{a}_x + (x^2z - 2xy)\mathbf{a}_y + x^2y\mathbf{a}_z$ C/m^2 at $P_A(2, 3, -1)$; (*b*) $\mathbf{D} = 2\rho z^2\sin^2\phi\,\mathbf{a}_\rho + \rho z^2\sin 2\phi\,\mathbf{a}_\phi + 2\rho^2 z\sin^2\phi\,\mathbf{a}_z$ C/m^2 at $P_B(\rho = 2, \phi = 110^\circ, z = -1)$; (*c*) $\mathbf{D} = 2r\sin\theta\cos\phi\,\mathbf{a}_r + r\cos\theta\cos\phi\,\mathbf{a}_\theta - r\sin\phi\,\mathbf{a}_\phi$ C/m^2 at $P_C(r = 1.5, \theta = 30^\circ, \phi = 50^\circ)$.

Ans. -10.00; 9.06; 1.29

Finally, we can combine Eqs. (9) and (12) and form the relation between electric flux density and charge density:

$$\text{div}\,\mathbf{D} = \rho_v \tag{15}$$

This is the first of Maxwell's four equations as they apply to electrostatics and steady magnetic fields, and it states that the electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there. This equation is aptly called the *point form of Gauss's law*. Gauss's law relates the flux leaving any closed surface to the charge enclosed, and Maxwell's first equation makes an identical statement on a per-unit-volume basis for a vanishingly small volume, or at a point. Because the divergence may be expressed as the sum of three partial derivatives, Maxwell's first equation is also described as the differential-equation form of Gauss's law, and conversely, Gauss's law is recognized as the integral form of Maxwell's first equation.

As a specific illustration, let us consider the divergence of **D** in the region about a point charge Q located at the origin. We have the field

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

and use (14), the expression for divergence in spherical coordinates:

$$\text{div }\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$

Because D_θ and D_ϕ are zero, we have

$$\text{div }\mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{Q}{4\pi r^2}\right) = 0 \qquad (\text{if } r \neq 0)$$

Thus, $\rho_\nu = 0$ everywhere except at the origin, where it is infinite.

The divergence operation is not limited to electric flux density; it can be applied to any vector field. We will apply it to several other electromagnetic fields in the coming chapters.

D3.8. Determine an expression for the volume charge density associated with each $\mathbf{D}$ field: (*a*) $\mathbf{D} = \frac{4xy}{z}\mathbf{a}_x + \frac{2x^2}{z}\mathbf{a}_y - \frac{2x^2y}{z^2}\mathbf{a}_z$; (*b*) $\mathbf{D} = z\sin\phi\,\mathbf{a}_\rho + z\cos\phi\,\mathbf{a}_\phi + \rho\sin\phi\,\mathbf{a}_z$; (*c*) $\mathbf{D} = \sin\theta\sin\phi\,\mathbf{a}_r + \cos\theta\sin\phi\,\mathbf{a}_\theta + \cos\phi\,\mathbf{a}_\phi$.

Ans. $\frac{4y}{z^3}(x^2+z^2)$; 0; 0.

3.6 THE VECTOR OPERATOR ∇ AND THE DIVERGENCE THEOREM

If we remind ourselves again that divergence is an operation on a vector yielding a scalar result, just as the dot product of two vectors gives a scalar result, it seems possible that we can find something that may be dotted formally with **D** to yield the scalar

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Obviously, this cannot be accomplished by using a dot *product*; the process must be a dot *operation*.

With this in mind, we define the *del operator* ∇ as a *vector operator*,

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z \qquad (16)$$

Similar *scalar operators* appear in several methods of solving differential equations where we often let D replace d/dx, D^2 replace d^2/dx^2, and so forth.[4] We agree on defining ∇ that it shall be treated in every way as an ordinary vector with the one important exception that partial derivatives result instead of products of scalars.

Consider $\nabla\cdot\mathbf{D}$, signifying

$$\nabla\cdot\mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right)\cdot(D_x\mathbf{a}_x + D_y\mathbf{a}_y + D_z\mathbf{a}_z)$$

[4] This scalar operator D, which will not appear again, is not to be confused with the electric flux density.

We first consider the dot products of the unit vectors, discarding the six zero terms, and obtain the result that we recognize as the divergence of **D**:

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \text{div}(\mathbf{D})$$

The use of $\nabla \cdot \mathbf{D}$ is much more prevalent than that of div **D**, although both usages have their advantages. Writing $\nabla \cdot \mathbf{D}$ allows us to obtain simply and quickly the correct partial derivatives, but only in rectangular coordinates, as we will see. On the other hand, div **D** is an excellent reminder of the physical interpretation of divergence. We shall use the operator notation $\nabla \cdot \mathbf{D}$ from now on to indicate the divergence operation.

The vector operator ∇ is used not only with divergence, but also with several other very important operations that appear later. One of these is ∇u, where u is any scalar field, and leads to

$$\nabla u = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z \right) u = \frac{\partial u}{\partial x}\mathbf{a}_x + \frac{\partial u}{\partial y}\mathbf{a}_y + \frac{\partial u}{\partial z}\mathbf{a}_z$$

The ∇ operator does not have a specific form in other coordinate systems. If we are considering **D** in cylindrical coordinates, then $\nabla \cdot \mathbf{D}$ still indicates the divergence of **D**, or

$$\nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

where this expression has been taken from Section 3.5. We have no form for ∇ itself to help us obtain this sum of partial derivatives. This means that ∇u, as yet unnamed but easily written in rectangular coordinates, cannot be expressed by us at this time in cylindrical coordinates. Such an expression will be obtained when ∇u is defined in Chapter 4.

We close our discussion of divergence by presenting a theorem that will be needed several times in later chapters, the *divergence theorem*. This theorem applies to any vector field for which the appropriate partial derivatives exist, although it is easiest for us to develop it for the electric flux density. We have actually obtained it already and now have little more to do than point it out and name it, for starting from Gauss's law, we have

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v dv = \int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv$$

The first and last expressions constitute the divergence theorem,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv \qquad (17)$$

which may be stated as follows:

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

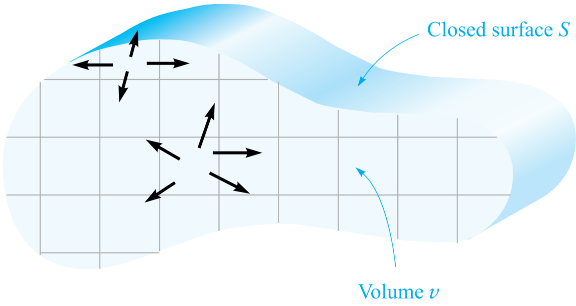

Figure 3.7 The divergence theorem states that the total flux crossing the closed surface is equal to the integral of the divergence of the flux density throughout the enclosed volume. The volume is shown here in cross section.

Again, we emphasize that the divergence theorem is true for any vector field, although we have obtained it specifically for the electric flux density **D**, and we will have occasion later to apply it to several different fields. Its benefits derive from the fact that it relates a triple integration *throughout some volume* to a double integration *over the surface* of that volume. For example, it is much easier to look for leaks in a bottle full of some agitated liquid by inspecting the surface than by calculating the velocity at every internal point.

The divergence theorem becomes obvious physically if we consider a volume v, shown in cross section in Figure 3.7, which is surrounded by a closed surface S. Division of the volume into a number of small compartments of differential size and consideration of one cell show that the flux diverging from such a cell *enters*, or *converges* on, the adjacent cells unless the cell contains a portion of the outer surface. In summary, the divergence of the flux density throughout a volume leads, then, to the same result as determining the net flux crossing the enclosing surface.

EXAMPLE 3.5

Evaluate both sides of the divergence theorem for the field $\mathbf{D} = 2xy\mathbf{a}_x + x^2\mathbf{a}_y$ C/m^2 and the rectangular parellelepiped formed by the planes $x = 0$ and 1, $y = 0$ and 2, and $z = 0$ and 3.

Solution. Evaluating the surface integral first, we note that **D** is parallel to the surfaces at $z = 0$ and $z = 3$, so $\mathbf{D} \cdot d\mathbf{S} = 0$ there. For the remaining four surfaces we have

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_0^3 \int_0^2 (\mathbf{D})_{x=0} \cdot (-dy\,dz\,\mathbf{a}_x) + \int_0^3 \int_0^2 (\mathbf{D})_{x=1} \cdot (dy\,dz\,\mathbf{a}_x)$$
$$+ \int_0^3 \int_0^1 (\mathbf{D})_{y=0} \cdot (-dx\,dz\,\mathbf{a}_y) + \int_0^3 \int_0^1 (\mathbf{D})_{y=2} \cdot (dx\,dz\,\mathbf{a}_y)$$

$$= -\int_0^3 \int_0^2 (D_x)_{x=0} dy\, dz + \int_0^3 \int_0^2 (D_x)_{x=1} dy\, dz$$

$$- \int_0^3 \int_0^1 (D_y)_{y=0} dx\, dz + \int_0^3 \int_0^1 (D_y)_{y=2} dx\, dz$$

However, $(D_x)_{x=0} = 0$, and $(D_y)_{y=0} = (D_y)_{y=2}$, which leaves only

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_0^3 \int_0^2 (D_x)_{x=1} dy\, dz = \int_0^3 \int_0^2 2y\, dy\, dz$$

$$= \int_0^3 4\, dz = 12$$

Since

$$\nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) = 2y$$

the volume integral becomes

$$\int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv = \int_0^3 \int_0^2 \int_0^1 2y\, dx\, dy\, dz = \int_0^3 \int_0^2 2y\, dy\, dz$$

$$= \int_0^3 4\, dz = 12$$

and the check is accomplished. Remembering Gauss's law, we see that we have also determined that a total charge of 12 C lies within this parallelepiped.

D3.9. Given the field $\mathbf{D} = 6\rho \sin \frac{1}{2}\phi\, \mathbf{a}_\rho + 1.5\rho \cos \frac{1}{2}\phi\, \mathbf{a}_\phi$ C/m^2, evaluate both sides of the divergence theorem for the region bounded by $\rho = 2$, $\phi = 0$, $\phi = \pi$, $z = 0$, and $z = 5$.

Ans. 225; 225

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2. Plonsey, R., and R. E. Collin. *Principles and Applications of Electromagnetic Fields*. New York: McGraw-Hill, 1961. The level of this text is somewhat higher than the one we are reading now, but it is an excellent text to read next. Gauss's law appears in the second chapter.
3. Plonus, M. A. *Applied Electromagnetics*. New York: McGraw-Hill, 1978. This book contains rather detailed descriptions of many practical devices that illustrate electromagnetic applications. For example, see the discussion of xerography on pp. 95–98 as an electrostatics application.
4. Skilling, H. H. *Fundamentals of Electric Waves*. 2d ed. New York: John Wiley & Sons, 1948. The operations of vector calculus are well illustrated. Divergence is discussed on pp. 22 and 38. Chapter 1 is interesting reading.

5. Thomas, G. B., Jr., and R. L. Finney. (see Suggested References for Chapter 1). The divergence theorem is developed and illustrated from several different points of view on pp. 976–980.

CHAPTER 3 PROBLEMS

WWW Quizzes

3.1 Suppose that the Faraday concentric sphere experiment is performed in free space using a central charge at the origin, Q_1, and with hemispheres of radius a. A second charge Q_2 (this time a point charge) is located at distance R from Q_1, where $R >> a$. (*a*) What is the force on the point charge before the hemispheres are assembled around Q_1? (*b*) What is the force on the point charge after the hemispheres are assembled but before they are discharged? (*c*) What is the force on the point charge after the hemispheres are assembled and after they are discharged? (*d*) Qualitatively, describe what happens as Q_2 is moved toward the sphere assembly to the extent that the condition $R >> a$ is no longer valid.

3.2 An electric field in free space is $\mathbf{E} = (5z^2/\epsilon_0)\,\hat{\mathbf{a}}_z$ V/m. Find the total charge contained within a cube, centered at the origin, of 4-m side length, in which all sides are parallel to coordinate axes (and therefore each side intersects an axis at ± 2).

3.3 The cylindrical surface $\rho = 8$ cm contains the surface charge density, $\rho_S = 5e^{-20|z|}$ nC/m^2. (*a*) What is the total amount of charge present? (*b*) How much electric flux leaves the surface $\rho = 8$ cm, 1 cm $< z <$ 5 cm, $30^\circ < \phi < 90^\circ$?

3.4 An electric field in free space is $\mathbf{E} = (5z^3/\epsilon_0)\,\hat{\mathbf{a}}_z$ V/m. Find the total charge contained within a sphere of 3-m radius, centered at the origin.

3.5 Let $\mathbf{D} = 4xy\mathbf{a}_x + 2(x^2 + z^2)\mathbf{a}_y + 4yz\mathbf{a}_z$ nC/m^2 and evaluate surface integrals to find the total charge enclosed in the rectangular parallelepiped $0 < x < 2$, $0 < y < 3$, $0 < z < 5$ m.

3.6 In free space, a volume charge of constant density $\rho_v = \rho_0$ exists within the region $-\infty < x < \infty$, $-\infty < y < \infty$, and $-d/2 < z < d/2$. Find **D** and **E** everywhere.

3.7 Volume charge density is located in free space as $\rho_v = 2e^{-1000r}$ nC/m^3 for $0 < r < 1$ mm, and $\rho_v = 0$ elsewhere. (*a*) Find the total charge enclosed by the spherical surface $r = 1$ mm. (*b*) By using Gauss's law, calculate the value of D_r on the surface $r = 1$ mm.

3.8 Use Gauss's law in integral form to show that an inverse distance field in spherical coordinates, $\mathbf{D} = A\mathbf{a}_r/r$, where A is a constant, requires every spherical shell of 1 m thickness to contain $4\pi A$ coulombs of charge. Does this indicate a continuous charge distribution? If so, find the charge density variation with r.

3.9 A uniform volume charge density of 80 C/m^3 is present throughout the region 8 mm $< r <$ 10 mm. Let $\rho_v = 0$ for $0 < r <$ 8 mm. (*a*) Find the total charge inside the spherical surface $r =$ 10 mm. (*b*) Find D_r at $r =$ 10 mm. (*c*) If there is no charge for $r >$ 10 mm, find D_r at $r =$ 20 mm.

3.10 An infinitely long cylindrical dielectric of radius b contains charge within its volume of density $\rho_v = a\rho^2$, where a is a constant. Find the electric field strength, **E**, both inside and outside the cylinder.

3.11 In cylindrical coordinates, let $\rho_v = 0$ for $\rho <$ 1 mm, $\rho_v = 2\sin(2000\pi\rho)$ nC/m^3 for 1 mm $< \rho <$ 1.5 mm, and $\rho_v = 0$ for $\rho >$ 1.5 mm. Find **D** everywhere.

3.12 The sun radiates a total power of about 3.86×10^{26} watts (W). If we imagine the sun's surface to be marked off in latitude and longitude and assume uniform radiation, (*a*) what power is radiated by the region lying between latitude 50° N and 60° N and longitude 12° W and 27° W? (*b*) What is the power density on a spherical surface 93,000,000 miles from the sun in W/m^2?

3.13 Spherical surfaces at $r =$ 2, 4, and 6 m carry uniform surface charge densities of 20 nC/m^2, -4 nC/m^2, and ρ_{S0}, respectively. (*a*) Find **D** at $r =$ 1, 3, and 5 m. (*b*) Determine ρ_{S0} such that $\mathbf{D} = 0$ at $r =$ 7 m.

3.14 A certain light-emitting diode (LED) is centered at the origin with its surface in the xy plane. At far distances, the LED appears as a point, but the glowing surface geometry produces a far-field radiation pattern that follows a raised cosine law: that is, the optical power (flux) density in watts/m^2 is given in spherical coordinates by

$$\mathbf{P}_d = P_0 \frac{\cos^2\theta}{2\pi r^2}\,\mathbf{a}_r \qquad \text{watts/m}^2$$

where θ is the angle measured with respect to the direction that is normal to the LED surface (in this case, the z axis), and r is the radial distance from the origin at which the power is detected. (*a*) In terms of P_0, find the total power in watts emitted in the upper half-space by the LED; (*b*) Find the cone angle, θ_1, within which half the total power is radiated, that is, within the range $0 < \theta < \theta_1$; (*c*) An optical detector, having a 1-mm^2 cross-sectional area, is positioned at $r =$ 1 m and at $\theta = 45°$, such that it faces the LED. If one milliwatt is measured by the detector, what (to a very good estimate) is the value of P_0?

3.15 Volume charge density is located as follows: $\rho_v = 0$ for $\rho <$ 1 mm and for $\rho >$ 2 mm, $\rho_v = 4\rho$ C/m^3 for $1 < \rho < 2$ mm. (*a*) Calculate the total charge in the region $0 < \rho < \rho_1$, $0 < z < L$, where $1 < \rho_1 < 2$ mm. (*b*) Use Gauss's law to determine D_ρ at $\rho = \rho_1$. (*c*) Evaluate D_ρ at $\rho =$ 0.8 mm, 1.6 mm, and 2.4 mm.

3.16 An electric flux density is given by $\mathbf{D} = D_0\,\mathbf{a}_\rho$, where D_0 is a given constant. (*a*) What charge density generates this field? (*b*) For the specified field, what total charge is contained within a cylinder of radius a and height b, where the cylinder axis is the z axis?

3.17 A cube is defined by $1 < x, y, z < 1.2$. If $\mathbf{D} = 2x^2y\mathbf{a}_x + 3x^2y^2\mathbf{a}_y$ C/m^2 (a) Apply Gauss's law to find the total flux leaving the closed surface of the cube. (b) Evaluate $\nabla \cdot \mathbf{D}$ at the center of the cube. (c) Estimate the total charge enclosed within the cube by using Eq. (8).

3.18 State whether the divergence of the following vector fields is positive, negative, or zero: (a) the thermal energy flow in J/(m^2 − s) at any point in a freezing ice cube; (b) the current density in A/m^2 in a bus bar carrying direct current; (c) the mass flow rate in kg/(m^2 − s) below the surface of water in a basin, in which the water is circulating clockwise as viewed from above.

3.19 A spherical surface of radius 3 mm is centered at $P(4, 1, 5)$ in free space. Let $\mathbf{D} = x\mathbf{a}_x$ C/m^2. Use the results of Section 3.4 to estimate the net electric flux leaving the spherical surface.

3.20 A radial electric field distribution in free space is given in spherical coordinates as:

$$\mathbf{E}_1 = \frac{r\rho_0}{3\epsilon_0}\,\mathbf{a}_r \qquad (r \le a)$$

$$\mathbf{E}_2 = \frac{(2a^3 - r^3)\rho_0}{3\epsilon_0 r^2}\,\mathbf{a}_r \qquad (a \le r \le b)$$

$$\mathbf{E}_3 = \frac{(2a^3 - b^3)\rho_0}{3\epsilon_0 r^2}\,\mathbf{a}_r \qquad (r \ge b)$$

where ρ_0, a, and b are constants. (a) Determine the volume charge density in the entire region ($0 \le r \le \infty$) by the appropriate use of $\nabla \cdot \mathbf{D} = \rho_v$. ($b$) In terms of given parameters, find the total charge, Q, within a sphere of radius r where $r > b$.

3.21 Calculate $\nabla \cdot \mathbf{D}$ at the point specified if (a) $\mathbf{D} = (1/z^2)[10xyz\,\mathbf{a}_x + 5x^2z\,\mathbf{a}_y + (2z^3 - 5x^2y)\,\mathbf{a}_z]$ at $P(-2, 3, 5)$; (b) $\mathbf{D} = 5z^2\,\mathbf{a}_\rho + 10\rho z\,\mathbf{a}_z$ at $P(3, -45^\circ, 5)$; (c) $\mathbf{D} = 2r\sin\theta\sin\phi\,\mathbf{a}_r + r\cos\theta\sin\phi\,\mathbf{a}_\theta + r\cos\phi\,\mathbf{a}_\phi$ at $P(3, 45^\circ, -45^\circ)$.

3.22 (a) A flux density field is given as $\mathbf{F}_1 = 5\mathbf{a}_z$. Evaluate the outward flux of $\mathbf{F}_1$ through the hemispherical surface, $r = a$, $0 < \theta < \pi/2$, $0 < \phi < 2\pi$. (b) What simple observation would have saved a lot of work in part a? (c) Now suppose the field is given by $\mathbf{F}_2 = 5z\mathbf{a}_z$. Using the appropriate surface integrals, evaluate the net outward flux of $\mathbf{F}_2$ through the closed surface consisting of the hemisphere of part a and its circular base in the xy plane. (d) Repeat part c by using the divergence theorem and an appropriate volume integral.

3.23 (a) A point charge Q lies at the origin. Show that div $\mathbf{D}$ is zero everywhere except at the origin. (b) Replace the point charge with a uniform volume charge density ρ_{v0} for $0 < r < a$. Relate ρ_{v0} to Q and a so that the total charge is the same. Find div $\mathbf{D}$ everywhere.

3.24 In a region in free space, electric flux density is found to be

$$\mathbf{D} = \begin{cases} \rho_0(z+2d)\,\mathbf{a}_z \ \text{C/m}^2 & (-2d \le z \le 0) \\ -\rho_0(z-2d)\,\mathbf{a}_z \ \text{C/m}^2 & (0 \le z \le 2d) \end{cases}$$

Everywhere else, $\mathbf{D} = 0$. (*a*) Using $\nabla \cdot \mathbf{D} = \rho_v$, find the volume charge density as a function of position everywhere. (*b*) Determine the electric flux that passes through the surface defined by $z = 0,\ -a \le x \le a,\ -b \le y \le b$. (*c*) Determine the total charge contained within the region $-a \le x \le a$, $-b \le y \le b,\ -d \le z \le d$. (*d*) Determine the total charge contained within the region $-a \le x \le a,\ -b \le y \le b,\ 0 \le z \le 2d$.

3.25 Within the spherical shell, $3 < r < 4$ m, the electric flux density is given as $\mathbf{D} = 5(r-3)^3\,\mathbf{a}_r$ C/m^2. (*a*) What is the volume charge density at $r = 4$? (*b*) What is the electric flux density at $r = 4$? (*c*) How much electric flux leaves the sphere $r = 4$? (*d*) How much charge is contained within the sphere $r = 4$?

3.26 If we have a perfect gas of mass density ρ_m kg/m^3, and we assign a velocity **U** m/s to each differential element, then the mass flow rate is $\rho_m\mathbf{U}$ kg/(m^2 − s). Physical reasoning then leads to the *continuity equation*, $\nabla \cdot (\rho_m\mathbf{U}) = -\partial\rho_m/\partial t$. (*a*) Explain in words the physical interpretation of this equation. (*b*) Show that $\oint_s \rho_m\mathbf{U} \cdot d\mathbf{S} = -dM/dt$, where M is the total mass of the gas within the constant closed surface S, and explain the physical significance of the equation.

3.27 Let $\mathbf{D} = 5.00r^2\mathbf{a}_r$ mC/m^2 for $r \le 0.08$ m and $\mathbf{D} = 0.205\,\mathbf{a}_r/r^2$ C/m^2 for $r \ge 0.08$ m. (*a*) Find ρ_v for $r = 0.06$ m. (*b*) Find ρ_v for $r = 0.1$ m. (*c*) What surface charge density could be located at $r = 0.08$ m to cause $\mathbf{D} = 0$ for $r > 0.08$ m?

3.28 Repeat Problem 3.8, but use $\nabla \cdot \mathbf{D} = \rho_v$ and take an appropriate volume integral.

3.29 In the region of free space that includes the volume $2 < x, y, z < 3$, $\mathbf{D} = \frac{2}{z^2}(yz\,\mathbf{a}_x + xz\,\mathbf{a}_y - 2xy\,\mathbf{a}_z)$ C/m^2. (*a*) Evaluate the volume integral side of the divergence theorem for the volume defined here. (*b*) Evaluate the surface integral side for the corresponding closed surface.

3.30 (*a*) Use Maxwell's first equation, $\nabla \cdot \mathbf{D} = \rho_v$, to describe the variation of the electric field intensity with x in a region in which no charge density exists and in which a nonhomogeneous dielectric has a permittivity that increases exponentially with x. The field has an x component only; (*b*) repeat part (*a*), but with a radially directed electric field (spherical coordinates), in which again $\rho_v = 0$, but in which the permittivity *decreases* exponentially with r.

3.31 Given the flux density $\mathbf{D} = \frac{16}{r}\cos(2\theta)\,\mathbf{a}_\theta$ C/m^2, use two different methods to find the total charge within the region $1 < r < 2$ m, $1 < \theta < 2$ rad, $1 < \phi < 2$ rad.

CHAPTER 4

Energy and Potential

In Chapters 2 and 3 we became acquainted with Coulomb's law and its use in finding the electric field about several simple distributions of charge, and also with Gauss's law and its application in determining the field about some symmetrical charge arrangements. The use of Gauss's law was invariably easier for these highly symmetrical distributions because the problem of integration always disappeared when the proper closed surface was chosen.

However, if we had attempted to find a slightly more complicated field, such as that of two unlike point charges separated by a small distance, we would have found it impossible to choose a suitable gaussian surface and obtain an answer. Coulomb's law, however, is more powerful and enables us to solve problems for which Gauss's law is not applicable. The application of Coulomb's law is laborious, detailed, and often quite complex, the reason for this being precisely the fact that the electric field intensity, a vector field, must be found directly from the charge distribution. Three different integrations are needed in general, one for each component, and the resolution of the vector into components usually adds to the complexity of the integrals.

Certainly it would be desirable if we could find some as yet undefined scalar function with a single integration and then determine the electric field from this scalar by some simple straightforward procedure, such as differentiation.

This scalar function does exist and is known as the *potential* or *potential field*. We shall find that it has a very real physical interpretation and is more familiar to most of us than is the electric field which it will be used to find.

We should expect, then, to be equipped soon with a third method of finding electric fields—a single scalar integration, although not always as simple as we might wish, followed by a pleasant differentiation.

4.1 ENERGY EXPENDED IN MOVING A POINT CHARGE IN AN ELECTRIC FIELD

The electric field intensity was defined as the force on a unit test charge at that point at which we wish to find the value of this vector field. If we attempt to move the test charge against the electric field, we have to exert a force equal and opposite to that exerted by the field, and this requires us to expend energy or do work. If we wish to move the charge in the direction of the field, our energy expenditure turns out to be negative; we do not do the work, the field does.

Suppose we wish to move a charge Q a distance $d\mathbf{L}$ in an electric field $\mathbf{E}$. The force on Q arising from the electric field is

$$\boxed{\mathbf{F}_E = Q\mathbf{E}} \tag{1}$$

where the subscript reminds us that this force arises from the field. The component of this force in the direction $d\mathbf{L}$ which we must overcome is

$$F_{EL} = \mathbf{F} \cdot \mathbf{a}_L = Q\mathbf{E} \cdot \mathbf{a}_L$$

where $\mathbf{a}_L =$ a unit vector in the direction of $d\mathbf{L}$.

The force that we must apply is equal and opposite to the force associated with the field,

$$F_{\text{appl}} = -Q\mathbf{E} \cdot \mathbf{a}_L$$

and the expenditure of energy is the product of the force and distance. That is, the differential work done by an external source moving charge Q is $dW = -Q\mathbf{E} \cdot \mathbf{a}_L dL$, or

$$\boxed{dW = -Q\mathbf{E} \cdot d\mathbf{L}} \tag{2}$$

where we have replaced $\mathbf{a}_L dL$ by the simpler expression $d\mathbf{L}$.

This differential amount of work required may be zero under several conditions determined easily from Eq. (2). There are the trivial conditions for which $\mathbf{E}$, Q, or $d\mathbf{L}$ is zero, and a much more important case in which $\mathbf{E}$ and $d\mathbf{L}$ are perpendicular. Here the charge is moved always in a direction at right angles to the electric field. We can draw on a good analogy between the electric field and the gravitational field, where, again, energy must be expended to move against the field. Sliding a mass around with constant velocity on a frictionless surface is an effortless process if the mass is moved along a constant elevation contour; positive or negative work must be done in moving it to a higher or lower elevation, respectively.

Returning to the charge in the electric field, the work required to move the charge a finite distance must be determined by integrating,

Interactives

$$\boxed{W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}} \tag{3}$$

where the path must be specified before the integral can be evaluated. The charge is assumed to be at rest at both its initial and final positions.

This definite integral is basic to field theory, and we shall devote the following section to its interpretation and evaluation.

D4.1. Given the electric field $\mathbf{E} = \frac{1}{z^2}(8xyz\mathbf{a}_x + 4x^2z\mathbf{a}_y - 4x^2y\mathbf{a}_z)$ V/m, find the differential amount of work done in moving a 6-nC charge a distance of 2 m, starting at $P(2, -2, 3)$ and proceeding in the direction $\mathbf{a}_L = (a)\ -\frac{6}{7}\mathbf{a}_x + \frac{3}{7}\mathbf{a}_y + \frac{2}{7}\mathbf{a}_z$; $(b)\ \frac{6}{7}\mathbf{a}_x - \frac{3}{7}\mathbf{a}_y - \frac{2}{7}\mathbf{a}_z$; $(c)\ \frac{3}{7}\mathbf{a}_x + \frac{6}{7}\mathbf{a}_y$.

Ans. −149.3 fJ; 149.3 fJ; 0

4.2 THE LINE INTEGRAL

The integral expression for the work done in moving a point charge Q from one position to another, Eq. (3), is an example of a line integral, which in vector-analysis notation always takes the form of the integral along some prescribed path of the dot product of a vector field and a differential vector path length $d\mathbf{L}$. Without using vector analysis we should have to write

$$W = -Q \int_{\text{init}}^{\text{final}} E_L \, dL$$

where E_L = component of $\mathbf{E}$ along $d\mathbf{L}$.

A line integral is like many other integrals which appear in advanced analysis, including the surface integral appearing in Gauss's law, in that it is essentially descriptive. We like to look at it much more than we like to work it out. It tells us to choose a path, break it up into a large number of very small segments, multiply the component of the field along each segment by the length of the segment, and then add the results for all the segments. This is a summation, of course, and the integral is obtained exactly only when the number of segments becomes infinite.

This procedure is indicated in Figure 4.1, where a path has been chosen from an initial position B to a final position[1] A and a *uniform electric field* is selected for simplicity. The path is divided into six segments, $\Delta\mathbf{L}_1, \Delta\mathbf{L}_2, \ldots, \Delta\mathbf{L}_6$, and the components of $\mathbf{E}$ along each segment are denoted by $E_{L1}, E_{L2}, \ldots, E_{L6}$. The work involved in moving a charge Q from B to A is then approximately

$$W = -Q(E_{L1}\Delta L_1 + E_{L2}\Delta L_2 + \cdots + E_{L6}\Delta L_6)$$

or, using vector notation,

$$W = -Q(\mathbf{E}_1 \cdot \Delta\mathbf{L}_1 + \mathbf{E}_2 \cdot \Delta\mathbf{L}_2 + \cdots + \mathbf{E}_6 \cdot \Delta\mathbf{L}_6)$$

[1] The final position is given the designation A to correspond with the convention for potential difference, as discussed in the following section.

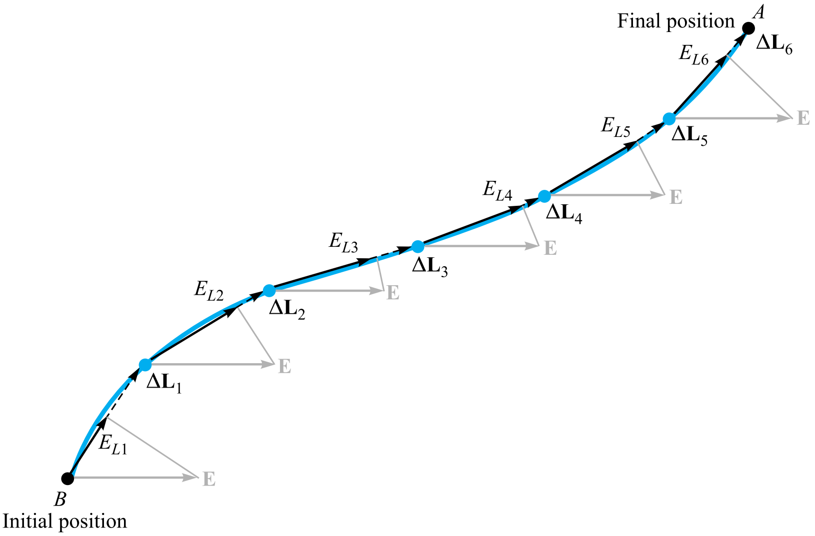

Figure 4.1 A graphical interpretation of a line integral in a uniform field. The line integral of **E** between points *B* and *A* is independent of the path selected, even in a nonuniform field; this result is not, in general, true for time-varying fields.

and because we have assumed a uniform field,

$$\mathbf{E}_1 = \mathbf{E}_2 = \cdots = \mathbf{E}_6$$
$$W = -Q\mathbf{E}\cdot(\Delta\mathbf{L}_1 + \Delta\mathbf{L}_2 + \cdots + \Delta\mathbf{L}_6)$$

What is this sum of vector segments in the preceding parentheses? Vectors add by the parallelogram law, and the sum is just the vector directed from the initial point B to the final point A, $\mathbf{L}_{BA}$. Therefore

$$W = -Q\mathbf{E}\cdot\mathbf{L}_{BA} \qquad (\text{uniform } \mathbf{E}) \tag{4}$$

Remembering the summation interpretation of the line integral, this result for the uniform field can be obtained rapidly now from the integral expression

$$W = -Q\int_B^A \mathbf{E}\cdot d\mathbf{L} \tag{5}$$

as applied to a uniform field

$$W = -Q\mathbf{E}\cdot\int_B^A d\mathbf{L}$$

where the last integral becomes $\mathbf{L}_{BA}$ and

$$W = -Q\mathbf{E}\cdot\mathbf{L}_{BA} \qquad (\text{uniform } \mathbf{E})$$

For this special case of a uniform electric field intensity, we should note that the work involved in moving the charge depends only on Q, $\mathbf{E}$, and $\mathbf{L}_{BA}$, a vector drawn from the initial to the final point of the path chosen. It does not depend on the particular path we have selected along which to carry the charge. We may proceed from B to A on a straight line or via the Old Chisholm Trail; the answer is the same. We show in Section 4.5 that an identical statement may be made for any nonuniform (static) $\mathbf{E}$ field.

Let us use several examples to illustrate the mechanics of setting up the line integral appearing in Eq. (5).

EXAMPLE 4.1

We are given the nonuniform field

$$\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z$$

and we are asked to determine the work expended in carrying 2C from $B(1, 0, 1)$ to $A(0.8, 0.6, 1)$ along the shorter arc of the circle

$$x^2 + y^2 = 1 \quad z = 1$$

Solution. We use $W = -Q\int_B^A \mathbf{E}\cdot d\mathbf{L}$, where $\mathbf{E}$ is not necessarily constant. Working in rectangular coordinates, the differential path $d\mathbf{L}$ is $dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$, and the integral becomes

$$\begin{aligned} W &= -Q\int_B^A \mathbf{E}\cdot d\mathbf{L} \\ &= -2\int_B^A (y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z)\cdot(dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z) \\ &= -2\int_1^{0.8} y\,dx - 2\int_0^{0.6} x\,dy - 4\int_1^1 dz \end{aligned}$$

where the limits on the integrals have been chosen to agree with the initial and final values of the appropriate variable of integration. Using the equation of the circular path (and selecting the sign of the radical which is correct for the quadrant involved), we have

$$\begin{aligned} W &= -2\int_1^{0.8} \sqrt{1-x^2}\,dx - 2\int_0^{0.6} \sqrt{1-y^2}\,dy - 0 \\ &= -\left[x\sqrt{1-x^2} + \sin^{-1} x\right]_1^{0.8} - \left[y\sqrt{1-y^2} + \sin^{-1} y\right]_0^{0.6} \\ &= -(0.48 + 0.927 - 0 - 1.571) - (0.48 + 0.644 - 0 - 0) \\ &= -0.96\,\text{J} \end{aligned}$$

EXAMPLE 4.2

Again find the work required to carry 2C from B to A in the same field, but this time use the straight-line path from B to A.

Solution. We start by determining the equations of the straight line. Any two of the following three equations for planes passing through the line are sufficient to define the line:

$$y - y_B = \frac{y_A - y_B}{x_A - x_B}(x - x_B)$$

$$z - z_B = \frac{z_A - z_B}{y_A - y_B}(y - y_B)$$

$$x - x_B = \frac{x_A - x_B}{z_A - z_B}(z - z_B)$$

From the first equation we have

$$y = -3(x - 1)$$

and from the second we obtain

$$z = 1$$

Thus,

$$W = -2\int_1^{0.8} y\,dx - 2\int_0^{0.6} x\,dy - 4\int_1^1 dz$$
$$= 6\int_1^{0.8} (x-1)\,dx - 2\int_0^{0.6}\left(1 - \frac{y}{3}\right)dy$$
$$= -0.96 \text{ J}$$

This is the same answer we found using the circular path between the same two points, and it again demonstrates the statement (unproved) that the work done is independent of the path taken in any electrostatic field.

It should be noted that the equations of the straight line show that $dy = -3\,dx$ and $dx = -\frac{1}{3}\,dy$. These substitutions may be made in the first two integrals, along with a change in limits, and the answer may be obtained by evaluating the new integrals. This method is often simpler if the integrand is a function of only one variable.

Note that the expressions for $d\mathbf{L}$ in our three coordinate systems use the differential lengths obtained in Chapter 1 (rectangular in Section 1.3, cylindrical in Section 1.8, and spherical in Section 1.9):

$$d\mathbf{L} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z \quad \text{(rectangular)} \tag{6}$$

$$d\mathbf{L} = d\rho\,\mathbf{a}_\rho + \rho\,d\phi\mathbf{a}_\phi + dz\,\mathbf{a}_z \quad \text{(cylindrical)} \tag{7}$$

$$d\mathbf{L} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi \quad \text{(spherical)} \tag{8}$$

The interrelationships among the several variables in each expression are determined from the specific equations for the path.

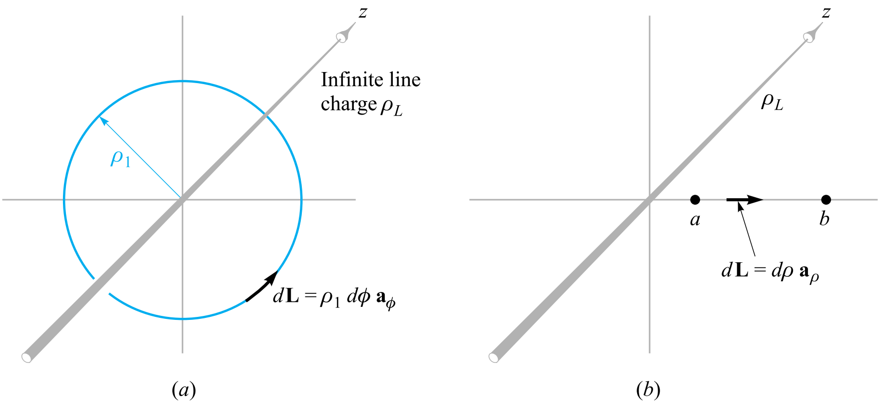

Figure 4.2 (*a*) A circular path and (*b*) a radial path along which a charge of *Q* is carried in the field of an infinite line charge. No work is expected in the former case.

As a final example illustrating the evaluation of the line integral, we investigate several paths that we might take near an infinite line charge. The field has been obtained several times and is entirely in the radial direction,

$$\mathbf{E} = E_\rho \mathbf{a}_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho}\mathbf{a}_\rho$$

First we find the work done in carrying the positive charge Q about a circular path of radius ρ_b centered at the line charge, as illustrated in Figure 4.2*a*. Without lifting a pencil, we see that the work must be nil, for the path is always perpendicular to the electric field intensity, or the force on the charge is always exerted at right angles to the direction in which we are moving it. For practice, however, we will set up the integral and obtain the answer.

The differential element $d\mathbf{L}$ is chosen in cylindrical coordinates, and the circular path selected demands that $d\rho$ and dz be zero, so $d\mathbf{L} = \rho_1\, d\phi\, \mathbf{a}_\phi$. The work is then

$$W = -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\epsilon_0\rho_1}\mathbf{a}_\rho \cdot \rho_1\, d\phi\, \mathbf{a}_\phi$$

$$= -Q\int_0^{2\pi} \frac{\rho_L}{2\pi\epsilon_0} d\phi\, \mathbf{a}_\rho \cdot \mathbf{a}_\phi = 0$$

We will now carry the charge from $\rho = a$ to $\rho = b$ along a radial path (Figure 4.2*b*). Here $d\mathbf{L} = d\rho\, \mathbf{a}_\rho$ and

$$W = -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho \cdot d\rho\, \mathbf{a}_\rho = -Q\int_a^b \frac{\rho_L}{2\pi\epsilon_0}\frac{d\rho}{\rho}$$

or

$$W = -\frac{Q\rho_L}{2\pi\epsilon_0}\ln\frac{b}{a}$$

Because b is larger than a, $\ln(b/a)$ is positive, and the work done is negative, indicating that the external source that is moving the charge receives energy.

One of the pitfalls in evaluating line integrals is a tendency to use too many minus signs when a charge is moved in the direction of a *decreasing* coordinate value. This is taken care of completely by the limits on the integral, and no misguided attempt should be made to change the sign of $d\mathbf{L}$. Suppose we carry Q from b to a (Figure 4.2b). We still have $d\mathbf{L} = d\rho\, \mathbf{a}_\rho$ and show the different direction by recognizing $\rho = b$ as the initial point and $\rho = a$ as the final point,

$$W = -Q \int_b^a \frac{\rho_L}{2\pi\epsilon_0} \frac{d\rho}{\rho} = \frac{Q\rho_L}{2\pi\epsilon_0} \ln \frac{b}{a}$$

This is the negative of the previous answer and is obviously correct.

D4.2. Calculate the work done in moving a 4-C charge from $B(1, 0, 0)$ to $A(0, 2, 0)$ along the path $y = 2 - 2x$, $z = 0$ in the field $\mathbf{E} =$ (a) $5\mathbf{a}_x$ V/m; (b) $5x\mathbf{a}_x$ V/m; (c) $5x\mathbf{a}_x + 5y\mathbf{a}_y$ V/m.

Ans. 20 J; 10 J; −30 J

D4.3. We will see later that a time-varying $\mathbf{E}$ field need not be conservative. (If it is not conservative, the work expressed by Eq. (3) may be a function of the path used.) Let $\mathbf{E} = y\mathbf{a}_x$ V/m at a certain instant of time, and calculate the work required to move a 3-C charge from (1, 3, 5) to (2, 0, 3) along the straight-line segments joining: (a) (1, 3, 5) to (2, 3, 5) to (2, 0, 5) to (2, 0, 3); (b) (1, 3, 5) to (1, 3, 3) to (1, 0, 3) to (2, 0, 3).

Ans. −9 J; 0

4.3 DEFINITION OF POTENTIAL DIFFERENCE AND POTENTIAL

We are now ready to define a new concept from the expression for the work done by an external source in moving a charge Q from one point to another in an electric field $\mathbf{E}$, "Potential difference and work."

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

In much the same way as we defined the electric field intensity as the force on a *unit* test charge, we now define *potential difference* V as the work done (by an external source) in moving a *unit* positive charge from one point to another in an electric field,

$$\text{Potential difference} = V = -\int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L} \tag{9}$$

We have to agree on the direction of movement, and we do this by stating that V_{AB} signifies the potential difference between points A and B and is the work done in moving the unit charge from B (last named) to A (first named). Thus, in determining V_{AB}, B is the initial point and A is the final point. The reason for this somewhat peculiar definition will become clearer shortly, when it is seen that the initial point B is often taken at infinity, whereas the final point A represents the fixed position of the charge; point A is thus inherently more significant.

Potential difference is measured in joules per coulomb, for which the *volt* is defined as a more common unit, abbreviated as V. Hence the potential difference between points A and B is

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} \text{ V} \tag{10}$$

and V_{AB} is positive if work is done in carrying the positive charge from B to A.

From the line-charge example of Section 4.2 we found that the work done in taking a charge Q from $\rho = b$ to $\rho = a$ was

$$W = \frac{Q\rho_L}{2\pi\epsilon_0} \ln \frac{b}{a}$$

Thus, the potential difference between points at $\rho = a$ and $\rho = b$ is

$$V_{ab} = \frac{W}{Q} = \frac{\rho_L}{2\pi\epsilon_0} \ln \frac{b}{a} \tag{11}$$

We can try out this definition by finding the potential difference between points A and B at radial distances r_A and r_B from a point charge Q. Choosing an origin at Q,

$$\mathbf{E} = E_r \mathbf{a}_r = \frac{Q}{4\pi\epsilon_0 r^2} \mathbf{a}_r$$

and

$$d\mathbf{L} = dr \, \mathbf{a}_r$$

we have

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r_A} - \frac{1}{r_B} \right) \tag{12}$$

If $r_B > r_A$, the potential difference V_{AB} is positive, indicating that energy is expended by the external source in bringing the positive charge from r_B to r_A. This agrees with the physical picture showing the two like charges repelling each other.

It is often convenient to speak of the *potential*, or *absolute potential*, of a point, rather than the potential difference between two points, but this means only that we agree to measure every potential difference with respect to a specified reference point that we consider to have zero potential. Common agreement must be reached on the zero reference before a statement of the potential has any significance. A person having one hand on the deflection plates of a cathode-ray tube that are "at a potential of 50 V" and the other hand on the cathode terminal would probably be too shaken up to understand that the cathode is not the zero reference, but that all potentials in that circuit are customarily measured with respect to the metallic shield about the tube. The cathode may be several thousands of volts negative with respect to the shield.

Perhaps the most universal zero reference point in experimental or physical potential measurements is "ground," by which we mean the potential of the surface region of the earth itself. Theoretically, we usually represent this surface by an infinite plane at zero potential, although some large-scale problems, such as those involving propagation across the Atlantic Ocean, require a spherical surface at zero potential.

Another widely used reference "point" is infinity. This usually appears in theoretical problems approximating a physical situation in which the earth is relatively far removed from the region in which we are interested, such as the static field near the wing tip of an airplane that has acquired a charge in flying through a thunderhead, or the field inside an atom. Working with the *gravitational* potential field on earth, the zero reference is normally taken at sea level; for an interplanetary mission, however, the zero reference is more conveniently selected at infinity.

A cylindrical surface of some definite radius may occasionally be used as a zero reference when cylindrical symmetry is present and infinity proves inconvenient. In a coaxial cable the outer conductor is selected as the zero reference for potential. And, of course, there are numerous special problems, such as those for which a two-sheeted hyperboloid or an oblate spheroid must be selected as the zero-potential reference, but these need not concern us immediately.

If the potential at point A is V_A and that at B is V_B, then

$$\boxed{V_{AB} = V_A - V_B} \tag{13}$$

where we necessarily agree that V_A and V_B shall have the same zero reference point.

D4.4. An electric field is expressed in rectangular coordinates by $\mathbf{E} = 6x^2\mathbf{a}_x + 6y\mathbf{a}_y + 4\mathbf{a}_z$ V/m. Find: (*a*) V_{MN} if points M and N are specified by $M(2, 6, -1)$ and $N(-3, -3, 2)$; (*b*) V_M if $V = 0$ at $Q(4, -2, -35)$; (*c*) V_N if $V = 2$ at $P(1, 2, -4)$.

Ans. -139.0 V; -120.0 V; 19.0 V

4.4 THE POTENTIAL FIELD OF A POINT CHARGE

In Section 4.3 we found an expression Eq. (12) for the potential difference between two points located at $r = r_A$ and $r = r_B$ in the field of a point charge Q placed at the origin. How might we conveniently define a zero reference for potential? The simplest possibility is to let $V = 0$ at infinity. If we let the point at $r = r_B$ recede to infinity, the potential at r_A becomes

$$V_A = \frac{Q}{4\pi\epsilon_0 r_A}$$

or, as there is no reason to identify this point with the A subscript,

$$V = \frac{Q}{4\pi\epsilon_0 r} \tag{14}$$

This expression defines the potential at any point distant r from a point charge Q at the origin, the potential at infinite radius being taken as the zero reference. Returning to a physical interpretation, we may say that $Q/4\pi\epsilon_0 r$ joules of work must be done in carrying a unit charge from infinity to any point r meters from the charge Q.

A convenient method to express the potential without selecting a specific zero reference entails identifying r_A as r once again and letting $Q/4\pi\epsilon_0 r_B$ be a constant. Then

$$V = \frac{Q}{4\pi\epsilon_0 r} + C_1 \tag{15}$$

and C_1 may be selected so that $V = 0$ at any desired value of r. We could also select the zero reference indirectly by electing to let V be V_0 at $r = r_0$.

It should be noted that the *potential difference* between two points is not a function of C_1.

Equations (14) and (15) represent the potential field of a point charge. The potential is a scalar field and does not involve any unit vectors.

We now define an *equipotential surface* as a surface composed of all those points having the same value of potential. All field lines would be perpendicular to such a surface at the points where they intersect it. Therefore, no work is involved in moving a unit charge around on an equipotential surface. The equipotential surfaces in the potential field of a point charge are spheres centered at the point charge.

Illustrations

An inspection of the form of the potential field of a point charge shows that it is an inverse-distance field, whereas the electric field intensity was found to be an inverse-square-law function. A similar result occurs for the gravitational force field of a point mass (inverse-square law) and the gravitational potential field (inverse distance). The gravitational force exerted by the earth on an object one million miles from it is four times that exerted on the same object two million miles away. The kinetic energy given to a freely falling object starting from the end of the universe with zero velocity, however, is only twice as much at one million miles as it is at two million miles.

D4.5. A 15-nC point charge is at the origin in free space. Calculate V_1 if point P_1 is located at $P_1(-2, 3, -1)$ and (*a*) $V = 0$ at $(6, 5, 4)$; (*b*) $V = 0$ at infinity; (*c*) $V = 5$ V at $(2, 0, 4)$.

Ans. 20.67 V; 36.0 V; 10.89 V

4.5 THE POTENTIAL FIELD OF A SYSTEM OF CHARGES: CONSERVATIVE PROPERTY

The potential at a point has been defined as the work done in bringing a unit positive charge from the zero reference to the point, and we have suspected that this work, and hence the potential, is independent of the path taken. If it were not, potential would not be a very useful concept.

Let us now prove our assertion. We do so by beginning with the potential field of the single point charge for which we showed, in Section 4.4, the independence with regard to the path, noting that the field is linear with respect to charge so that superposition is applicable. It will then follow that the potential of a system of charges has a value at any point which is independent of the path taken in carrying the test charge to that point.

Thus the potential field of a single point charge, which we shall identify as Q_1 and locate at $\mathbf{r}_1$, involves only the distance $|\mathbf{r} - \mathbf{r}_1|$ from Q_1 to the point at $\mathbf{r}$ where we are establishing the value of the potential. For a zero reference at infinity, we have

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1|}$$

The potential arising from two charges, Q_1 at $\mathbf{r}_1$ and Q_2 at $\mathbf{r}_2$, is a function only of $|\mathbf{r} - \mathbf{r}_1|$ and $|\mathbf{r} - \mathbf{r}_2|$, the distances from Q_1 and Q_2 to the field point, respectively.

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1|} + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_2|}$$

Continuing to add charges, we find that the potential arising from n point charges is

$$V(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_m|} \tag{16}$$

If each point charge is now represented as a small element of a continuous volume charge distribution $\rho_\nu \Delta\nu$, then

$$V(\mathbf{r}) = \frac{\rho_\nu(\mathbf{r}_1)\Delta\nu_1}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1|} + \frac{\rho_\nu(\mathbf{r}_2)\Delta\nu_2}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_2|} + \cdots + \frac{\rho_\nu(\mathbf{r}_n)\Delta\nu_n}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_n|}$$

As we allow the number of elements to become infinite, we obtain the integral expression

$$V(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_\nu(\mathbf{r}')\,dv'}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|} \tag{17}$$

We have come quite a distance from the potential field of the single point charge, and it might be helpful to examine Eq. (17) and refresh ourselves as to the meaning of each term. The potential $V(\mathbf{r})$ is determined with respect to a zero reference potential at infinity and is an exact measure of the work done in bringing a unit charge from infinity to the field point at $\mathbf{r}$ where we are finding the potential. The volume charge density $\rho_v(\mathbf{r}')$ and differential volume element dv' combine to represent a differential amount of charge $\rho_v(\mathbf{r}')\,dv'$ located at $\mathbf{r}'$. The distance $|\mathbf{r}-\mathbf{r}'|$ is that distance from the source point to the field point. The integral is a multiple (volume) integral.

If the charge distribution takes the form of a line charge or a surface charge, the integration is along the line or over the surface:

$$V(\mathbf{r}) = \int \frac{\rho_L(\mathbf{r}')\,dL'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|} \tag{18}$$

$$V(\mathbf{r}) = \int_S \frac{\rho_S(\mathbf{r}')\,dS'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|} \tag{19}$$

The most general expression for potential is obtained by combining Eqs. (16)–(19).

These integral expressions for potential in terms of the charge distribution should be compared with similar expressions for the electric field intensity, such as Eq. (15) in Section 2.3:

$$\mathbf{E}(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_v(\mathbf{r}')\,dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\,\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

The potential again is inverse distance, and the electric field intensity, inverse-square law. The latter, of course, is also a vector field.

EXAMPLE 4.3

To illustrate the use of one of these potential integrals, we will find V on the z axis for a uniform line charge ρ_L in the form of a ring, $\rho = a$, in the $z = 0$ plane, as shown in Figure 4.3.

Solution. Working with Eq. (18), we have $dL' = a\,d\phi'$, $\mathbf{r} = z\mathbf{a}_z$, $\mathbf{r}' = a\mathbf{a}_\rho$, $|\mathbf{r}-\mathbf{r}'| = \sqrt{a^2+z^2}$, and

$$V = \int_0^{2\pi} \frac{\rho_L a\,d\phi'}{4\pi\epsilon_0\sqrt{a^2+z^2}} = \frac{\rho_L a}{2\epsilon_0\sqrt{a^2+z^2}}$$

For a zero reference at infinity, then:

1. The potential arising from a single point charge is the work done in carrying a unit positive charge from infinity to the point at which we desire the potential, and the work is independent of the path chosen between those two points.
2. The potential field in the presence of a number of point charges is the sum of the individual potential fields arising from each charge.
3. The potential arising from a number of point charges or any continuous charge distribution may therefore be found by carrying a unit charge from infinity to the point in question along any path we choose.

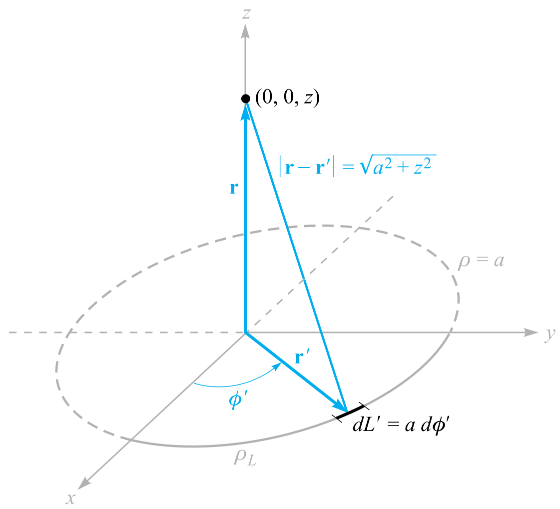

Figure 4.3 The potential field of a ring of uniform line charge density is easily obtained from $V = \int \rho_L(\mathbf{r}')\,dL'/(4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|)$.

In other words, the expression for potential (zero reference at infinity),

$$V_A = -\int_{\infty}^{A} \mathbf{E} \cdot d\mathbf{L}$$

or potential difference,

$$V_{AB} = V_A - V_B = -\int_{B}^{A} \mathbf{E} \cdot d\mathbf{L}$$

is not dependent on the path chosen for the line integral, regardless of the source of the **E** field.

This result is often stated concisely by recognizing that no work is done in carrying the unit charge around any *closed path*, or

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0 \tag{20}$$

A small circle is placed on the integral sign to indicate the closed nature of the path. This symbol also appeared in the formulation of Gauss's law, where a closed *surface* integral was used.

Equation (20) is true for *static* fields, but we will see in Chapter 9 that Faraday demonstrated it was incomplete when time-varying magnetic fields were present. One of Maxwell's greatest contributions to electromagnetic theory was in showing that a time-varying electric field produces a magnetic field, and therefore we should expect to find later that Eq. (20) is not correct when either **E** or the magnetic field varies with time.

Restricting our attention to the static case where **E** does not change with time, consider the dc circuit shown in Figure 4.4. Two points, A and B, are marked, and

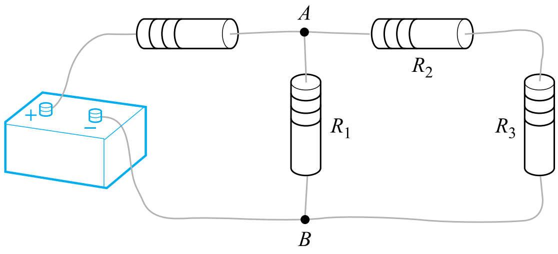

Figure 4.4 A simple dc-circuit problem that must be solved by applying $\oint \mathbf{E} \cdot d\mathbf{L} = 0$ in the form of Kirchhoff's voltage law.

(20) states that no work is involved in carrying a unit charge from A through R_2 and R_3 to B and back to A through R_1, or that the sum of the potential differences around any closed path is zero.

Equation (20) is therefore just a more general form of Kirchhoff's circuital law for voltages, more general in that we can apply it to any region where an electric field exists and we are not restricted to a conventional circuit composed of wires, resistances, and batteries. Equation (20) must be amended before we can apply it to time-varying fields.

Any field that satisfies an equation of the form of Eq. (20), (i.e., where the closed line integral of the field is zero) is said to be a *conservative field*. The name arises from the fact that no work is done (or that energy is *conserved*) around a closed path. The gravitational field is also conservative, for any energy expended in moving (raising) an object against the field is recovered exactly when the object is returned (lowered) to its original position. A nonconservative gravitational field could solve our energy problems forever.

Given a *nonconservative* field, it is of course possible that the line integral may be zero for certain closed paths. For example, consider the force field, $\mathbf{F} = \sin \pi\rho \, \mathbf{a}_\phi$. Around a circular path of radius $\rho = \rho_1$, we have $d\mathbf{L} = \rho \, d\phi \, \mathbf{a}_\phi$, and

$$\oint \mathbf{F} \cdot d\mathbf{L} = \int_0^{2\pi} \sin \pi\rho_1 \mathbf{a}_\phi \cdot \rho_1 d\phi \, \mathbf{a}_\phi = \int_0^{2\pi} \rho_1 \sin \pi\rho_1 \, d\phi$$
$$= 2\pi\rho_1 \sin \pi\rho_1$$

The integral is zero if $\rho_1 = 1, 2, 3, \ldots,$ etc., but it is not zero for other values of ρ_1, or for most other closed paths, and the given field is not conservative. A conservative field must yield a zero value for the line integral around every possible closed path.

D4.6. If we take the zero reference for potential at infinity, find the potential at (0, 0, 2) caused by this charge configuration in free space (a) 12 nC/m on the line $\rho = 2.5$ m, $z = 0$; (b) point charge of 18 nC at (1, 2, −1); (c) 12 nC/m on the line $y = 2.5$, $z = 0$, $-1.0 < x < 1.0$.

Ans. 529 V; 43.2 V; 66.3 V

4.6 POTENTIAL GRADIENT

Interactives

We now have two methods of determining potential, one directly from the electric field intensity by means of a line integral, and another from the basic charge distribution itself by a volume integral. Neither method is very helpful in determining the fields in most practical problems, however, for as we will see later, neither the electric field intensity nor the charge distribution is very often known. Preliminary information is much more apt to consist of a description of two equipotential surfaces, such as the statement that we have two parallel conductors of circular cross section at potentials of 100 and −100 V. Perhaps we wish to find the capacitance between the conductors, or the charge and current distribution on the conductors from which losses may be calculated.

These quantities may be easily obtained from the potential field, and our immediate goal will be a simple method of finding the electric field intensity from the potential.

We already have the general line-integral relationship between these quantities,

$$V = -\int \mathbf{E} \cdot d\mathbf{L} \tag{21}$$

but this is much easier to use in the reverse direction: given **E**, find V.

However, Eq. (21) may be applied to a very short element of length $\Delta\mathbf{L}$ along which **E** is essentially constant, leading to an incremental potential difference ΔV,

$$\Delta V \doteq -\mathbf{E} \cdot \Delta\mathbf{L} \tag{22}$$

Now consider a general region of space, as shown in Figure 4.5, in which **E** and V both change as we move from point to point. Equation (22) tells us to choose an incremental vector element of length $\Delta\mathbf{L} = \Delta L\, \mathbf{a}_L$ and multiply its magnitude by

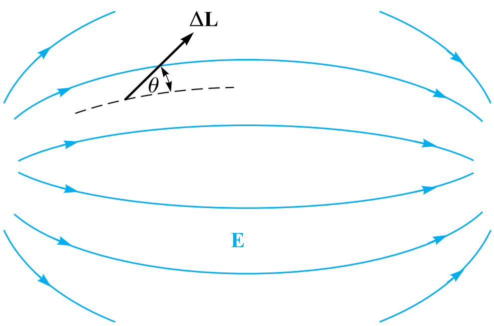

Figure 4.5 A vector incremental element of length ΔL is shown making an angle of θ with an **E** field, indicated by its streamlines. The sources of the field are not shown.

the component of **E** in the direction of $\mathbf{a}_L$ (one interpretation of the dot product) to obtain the small potential difference between the final and initial points of $\Delta\mathbf{L}$.

If we designate the angle between $\Delta\mathbf{L}$ and **E** as θ, then

$$\Delta V \doteq -E\Delta L \cos\theta$$

We now pass to the limit and consider the derivative dV/dL. To do this, we need to show that V may be interpreted as a *function* $V(x, y, z)$. So far, V is merely the result of the line integral (21). If we assume a specified starting point or zero reference and then let our end point be (x, y, z), we know that the result of the integration is a unique function of the end point (x, y, z) because **E** is a conservative field. Therefore V is a single-valued function $V(x, y, z)$. We may then pass to the limit and obtain

$$\frac{dV}{dL} = -E\cos\theta$$

In which direction should $\Delta\mathbf{L}$ be placed to obtain a maximum value of ΔV? Remember that **E** is a definite value at the point at which we are working and is independent of the direction of $\Delta\mathbf{L}$. The magnitude ΔL is also constant, and our variable is $\mathbf{a}_L$, the unit vector showing the direction of $\Delta\mathbf{L}$. It is obvious that the maximum positive increment of potential, ΔV_{max}, will occur when $\cos\theta$ is -1, or $\Delta\mathbf{L}$ points in the direction *opposite* to **E**. For this condition,

$$\left.\frac{dV}{dL}\right|_{\text{max}} = E$$

This little exercise shows us two characteristics of the relationship between **E** and V at any point:

1. The magnitude of the electric field intensity is given by the maximum value of the rate of change of potential with distance.
2. This maximum value is obtained when the direction of the distance increment is opposite to **E** or, in other words, the direction of **E** is *opposite* to the direction in which the potential is *increasing* the most rapidly.

We now illustrate these relationships in terms of potential. Figure 4.6 is intended to show the information we have been given about some potential field. It does this by showing the equipotential surfaces (shown as lines in the two-dimensional sketch). We desire information about the electric field intensity at point P. Starting at P, we lay off a small incremental distance $\Delta\mathbf{L}$ in various directions, hunting for that direction in which the potential is changing (increasing) the most rapidly. From the sketch, this direction appears to be left and slightly upward. From our second characteristic above, the electric field intensity is therefore oppositely directed, or to the right and slightly downward at P. Its magnitude is given by dividing the small increase in potential by the small element of length.

It seems likely that the direction in which the potential is increasing the most rapidly is perpendicular to the equipotentials (in the direction of *increasing* potential), and this is correct, for if $\Delta\mathbf{L}$ is directed along an equipotential, $\Delta V = 0$ by our

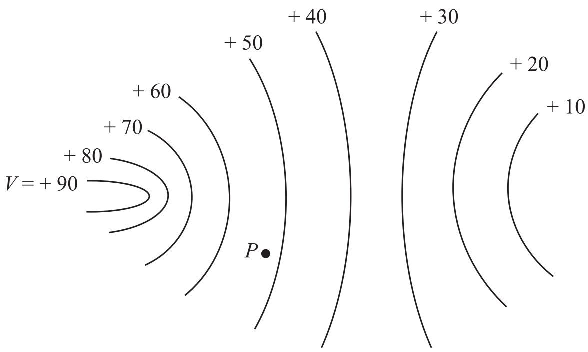

Figure 4.6 A potential field is shown by its equipotential surfaces. At any point the **E** field is normal to the equipotential surface passing through that point and is directed toward the more negative surfaces.

definition of an equipotential surface. But then

$$\Delta V = -\mathbf{E} \cdot \Delta \mathbf{L} = 0$$

and as neither $\mathbf{E}$ nor $\Delta\mathbf{L}$ is zero, $\mathbf{E}$ must be perpendicular to this $\Delta\mathbf{L}$ or perpendicular to the equipotentials.

Because the potential field information is more likely to be determined first, let us describe the direction of $\Delta\mathbf{L}$, which leads to a maximum increase in potential mathematically in terms of the potential field rather than the electric field intensity. We do this by letting $\mathbf{a}_N$ be a unit vector normal to the equipotential surface and directed toward the higher potentials. The electric field intensity is then expressed in terms of the potential,

$$\mathbf{E} = -\left.\frac{dV}{dL}\right|_{max} \mathbf{a}_N \tag{23}$$

which shows that the magnitude of $\mathbf{E}$ is given by the maximum space rate of change of V and the direction of $\mathbf{E}$ is *normal* to the equipotential surface (in the direction of *decreasing* potential).

Because $dV/dL|_{max}$ occurs when $\Delta\mathbf{L}$ is in the direction of $\mathbf{a}_N$, we may remind ourselves of this fact by letting

$$\left.\frac{dV}{dL}\right|_{max} = \frac{dV}{dN}$$

and

$$\mathbf{E} = -\frac{dV}{dN}\mathbf{a}_N \tag{24}$$

Either Eq. (23) or Eq. (24) provides a physical interpretation of the process of finding the electric field intensity from the potential. Both are descriptive of a general procedure, and we do not intend to use them directly to obtain quantitative information.

This procedure leading from V to $\mathbf{E}$ is not unique to this pair of quantities, however, but has appeared as the relationship between a scalar and a vector field in hydraulics, thermodynamics, and magnetics, and indeed in almost every field to which vector analysis has been applied.

The operation on V by which $-\mathbf{E}$ is obtained is known as the *gradient*, and the gradient of a scalar field T is defined as

$$\text{Gradient of } T = \text{grad } T = \frac{dT}{dN}\mathbf{a}_N \tag{25}$$

where $\mathbf{a}_N$ is a unit vector normal to the equipotential surfaces, and that normal is chosen, which points in the direction of increasing values of T.

Using this new term, we now may write the relationship between V and $\mathbf{E}$ as

$$\mathbf{E} = -\text{grad } V \tag{26}$$

Because we have shown that V is a unique function of x, y, and z, we may take its total differential

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

But we also have

$$dV = -\mathbf{E} \cdot d\mathbf{L} = -E_x\, dx - E_y\, dy - E_z\, dz$$

Because both expressions are true for any dx, dy, and dz, then

$$E_x = -\frac{\partial V}{\partial x}$$

$$E_y = -\frac{\partial V}{\partial y}$$

$$E_z = -\frac{\partial V}{\partial z}$$

These results may be combined vectorially to yield

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right) \tag{27}$$

and comparing Eqs. (26) and (27) provides us with an expression which may be used to evaluate the gradient in rectangular coordinates,

$$\text{grad } V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \tag{28}$$

The gradient of a scalar is a vector, and old quizzes show that the unit vectors that are often incorrectly added to the divergence expression appear to be those that were incorrectly removed from the gradient. Once the physical interpretation of the gradient, expressed by Eq. (25), is grasped as showing the maximum space rate of change of a scalar quantity and *the direction in which this maximum occurs*, the vector nature of the gradient should be self-evident.

The vector operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

may be used formally as an operator on a scalar, T, ∇T, producing

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{a}_x + \frac{\partial T}{\partial y}\mathbf{a}_y + \frac{\partial T}{\partial z}\mathbf{a}_z$$

from which we see that

$$\nabla T = \text{grad}\, T$$

Interactives

This allows us to use a very compact expression to relate **E** and V,

$$\mathbf{E} = -\nabla V \tag{29}$$

The gradient may be expressed in terms of partial derivatives in other coordinate systems through the application of its definition Eq. (25). These expressions are derived in Appendix A and repeated here for convenience when dealing with problems having cylindrical or spherical symmetry. They also appear inside the back cover.

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \quad \text{(rectangular)} \tag{30}$$

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z \quad \text{(cylindrical)} \tag{31}$$

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi \quad \text{(spherical)} \tag{32}$$

Note that the denominator of each term has the form of one of the components of $d\mathbf{L}$ in that coordinate system, except that partial differentials replace ordinary differentials; for example, $r\sin\theta\, d\phi$ becomes $r\sin\theta\, \partial\phi$.

We now illustrate the gradient concept with an example.

EXAMPLE 4.4

Given the potential field, $V = 2x^2y - 5z$, and a point $P(-4, 3, 6)$, we wish to find several numerical values at point P: the potential V, the electric field intensity **E**, the direction of **E**, the electric flux density **D**, and the volume charge density ρ_v.

Solution. The potential at $P(-4, 5, 6)$ is

$$V_P = 2(-4)^2(3) - 5(6) = 66 \text{ V}$$

Next, we may use the gradient operation to obtain the electric field intensity,

$$\mathbf{E} = -\nabla V = -4xy\mathbf{a}_x - 2x^2\mathbf{a}_y + 5\mathbf{a}_z \text{ V/m}$$

The value of $\mathbf{E}$ at point P is

$$\mathbf{E}_P = 48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z \text{ V/m}$$

and

$$|\mathbf{E}_P| = \sqrt{48^2 + (-32)^2 + 5^2} = 57.9 \text{ V/m}$$

The direction of $\mathbf{E}$ at P is given by the unit vector

$$\begin{aligned}\mathbf{a}_{E,P} &= (48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z)/57.9 \\ &= 0.829\mathbf{a}_x - 0.553\mathbf{a}_y + 0.086\mathbf{a}_z\end{aligned}$$

If we assume these fields exist in free space, then

$$\mathbf{D} = \epsilon_0\mathbf{E} = -35.4xy\,\mathbf{a}_x - 17.71x^2\,\mathbf{a}_y + 44.3\,\mathbf{a}_z \text{ pC/m}^3$$

Finally, we may use the divergence relationship to find the volume charge density that is the source of the given potential field,

$$\rho_v = \nabla \cdot \mathbf{D} = -35.4y \text{ pC/m}^3$$

At P, $\rho_v = -106.2 \text{ pC/m}^3$.

D4.7. A portion of a two-dimensional ($E_z = 0$) potential field is shown in Figure 4.7. The grid lines are 1 mm apart in the actual field. Determine approximate values for $\mathbf{E}$ in rectangular coordinates at: (a) a; (b) b; (c) c.

Ans. $-1075\mathbf{a}_y$ V/m; $-600\mathbf{a}_x - 700\mathbf{a}_y$ V/m; $-500\mathbf{a}_x - 650\mathbf{a}_y$ V/m

D4.8. Given the potential field in cylindrical coordinates, $V = \frac{100}{z^2+1}\rho\cos\phi$ V, and point P at $\rho = 3$ m, $\phi = 60^\circ$, $z = 2$ m, find values at P for (a) V; (b) $\mathbf{E}$; (c) E; (d) dV/dN; (e) $\mathbf{a}_N$; (f) ρ_v in free space.

Ans. 30.0 V; $-10.00\mathbf{a}_\rho + 17.3\mathbf{a}_\phi + 24.0\mathbf{a}_z$ V/m; 31.2 V/m; 31.2 V/m; $0.32\mathbf{a}_\rho - 0.55\mathbf{a}_\phi - 0.77\mathbf{a}_z$; -234 pC/m^3

4.7 THE ELECTRIC DIPOLE

The dipole fields that we develop in this section are quite important because they form the basis for the behavior of dielectric materials in electric fields, as discussed in Chapter 6, as well as justifying the use of images, as described in Section 5.5 of Chapter 5. Moreover, this development will serve to illustrate the importance of the potential concept presented in this chapter.

An *electric dipole*, or simply a *dipole*, is the name given to two point charges of equal magnitude and opposite sign, separated by a distance that is small compared to

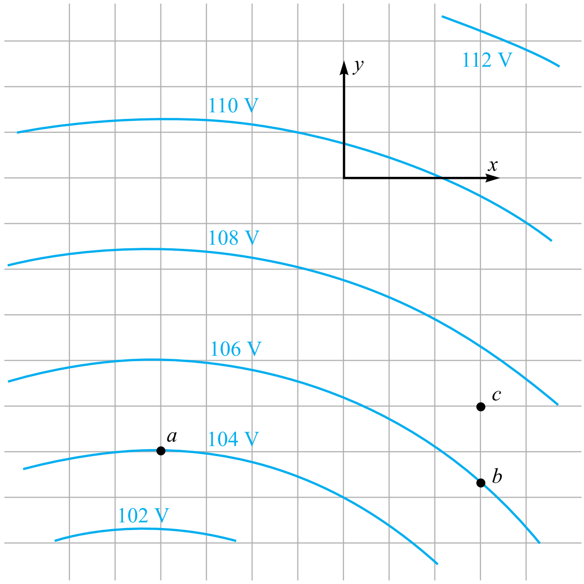

Figure 4.7 See Problem D4.7.

the distance to the point P at which we want to know the electric and potential fields. The dipole is shown in Figure 4.8a. The distant point P is described by the spherical coordinates r, θ, and $\phi = 90^\circ$, in view of the azimuthal symmetry. The positive and negative point charges have separation d and rectangular coordinates $(0, 0, \frac{1}{2}d)$ and $(0, 0, -\frac{1}{2}d)$, respectively.

So much for the geometry. What would we do next? Should we find the total electric field intensity by adding the known fields of each point charge? Would it be easier to find the total potential field first? In either case, having found one, we will find the other from it before calling the problem solved.

If we choose to find **E** first, we will have two components to keep track of in spherical coordinates (symmetry shows E_ϕ is zero), and then the only way to find V from **E** is by use of the line integral. This last step includes establishing a suitable zero reference for potential, since the line integral gives us only the potential difference between the two points at the ends of the integral path.

On the other hand, the determination of V first is a much simpler problem. This is because we find the potential as a function of position by simply adding the scalar potentials from the two charges. The position-dependent vector magnitude and direction of **E** are subsequently evaluated with relative ease by taking the negative gradient of V.

Choosing this simpler method, we let the distances from Q and $-Q$ to P be R_1 and R_2, respectively, and write the total potential as

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\epsilon_0}\frac{R_2 - R_1}{R_1 R_2}$$

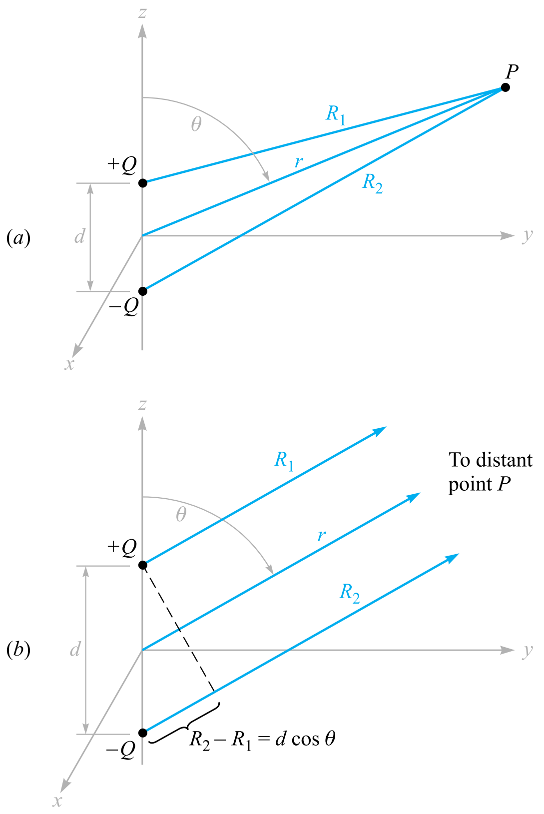

Figure 4.8 (*a*) The geometry of the problem of an electric dipole. The dipole moment $p = Qd$ is in the $\mathbf{a}_z$ direction. (*b*) For a distant point P, R_1 is essentially parallel to R_2, and we find that $R_2 - R_1 = d\cos\theta$.

Note that the plane $z = 0$, midway between the two point charges, is the locus of points for which $R_1 = R_2$, and is therefore at zero potential, as are all points at infinity.

For a distant point, $R_1 \doteq R_2$, and the R_1R_2 product in the denominator may be replaced by r^2. The approximation may not be made in the numerator, however, without obtaining the trivial answer that the potential field approaches zero as we go very far away from the dipole. Coming back a little closer to the dipole, we see from Figure 4.8*b* that $R_2 - R_1$ may be approximated very easily if R_1 and R_2 are assumed to be parallel,

$$R_2 - R_1 \doteq d\cos\theta$$

The final result is then

$$V = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2} \tag{33}$$

Again, we note that the plane $z = 0$ ($\theta = 90°$) is at zero potential.

Using the gradient relationship in spherical coordinates,

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi\right)$$

we obtain

$$\mathbf{E} = -\left(-\frac{Qd\cos\theta}{2\pi\epsilon_0 r^3}\mathbf{a}_r - \frac{Qd\sin\theta}{4\pi\epsilon_0 r^3}\mathbf{a}_\theta\right) \tag{34}$$

or

$$\mathbf{E} = \frac{Qd}{4\pi\epsilon_0 r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta) \tag{35}$$

These are the desired distant fields of the dipole, obtained with a very small amount of work. Any student who has several hours to spend may try to work the problem in the reverse direction—the authors consider the process too long and detailed to include here, even for effect.

To obtain a plot of the potential field, we choose a dipole such that $Qd/(4\pi\epsilon_0) = 1$, and then $\cos\theta = Vr^2$. The colored lines in Figure 4.9 indicate equipotentials for which $V = 0, +0.2, +0.4, +0.6, +0.8$, and $+1$, as indicated. The dipole axis is vertical, with the positive charge on the top. The streamlines for the electric field are obtained by applying the methods of Section 2.6 in spherical coordinates,

$$\frac{E_\theta}{E_r} = \frac{r\,d\theta}{dr} = \frac{\sin\theta}{2\cos\theta}$$

or

$$\frac{dr}{r} = 2\cot\theta\,d\theta$$

from which we obtain

$$r = C_1\sin^2\theta$$

The black streamlines shown in Figure 4.9 are for $C_1 = 1, 1.5, 2$, and 2.5.

The potential field of the dipole, Eq. (33), may be simplified by making use of the dipole moment. We first identify the vector length directed from $-Q$ to $+Q$ as $\mathbf{d}$

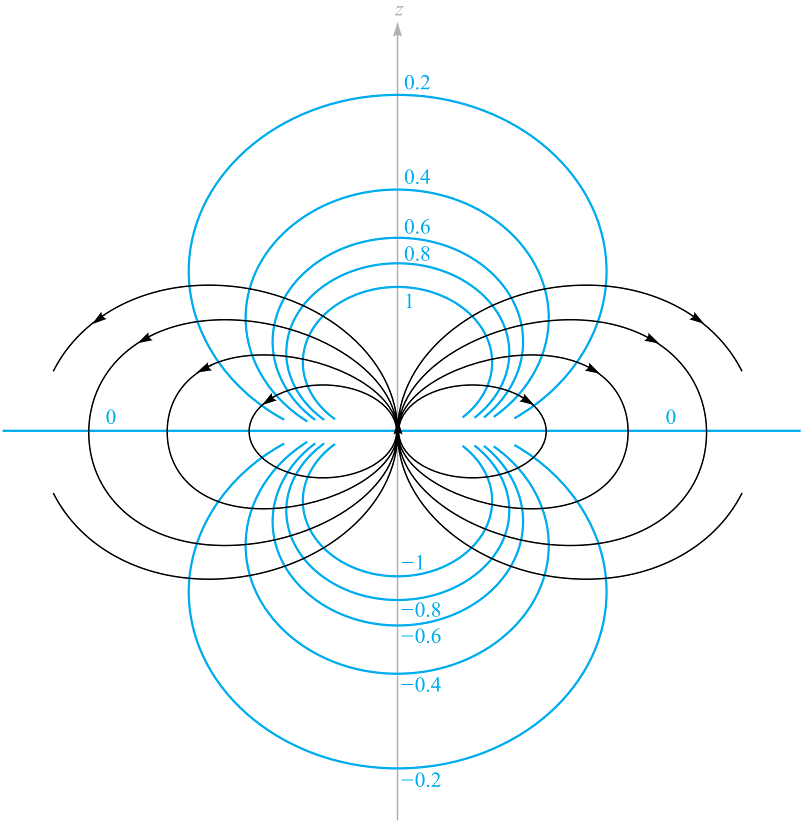

Figure 4.9 The electrostatic field of a point dipole with its moment in the $\mathbf{a}_z$ direction. Six equipotential surfaces are labeled with relative values of V.

and then define the *dipole moment* as $Q\mathbf{d}$ and assign it the symbol $\mathbf{p}$. Thus

$$\boxed{\mathbf{p} = Q\mathbf{d}} \tag{36}$$

The units of $\mathbf{p}$ are C · m.

Because $\mathbf{d} \cdot \mathbf{a}_r = d\cos\theta$, we then have

$$V = \frac{\mathbf{p} \cdot \mathbf{a}_r}{4\pi\epsilon_0 r^2} \tag{37}$$

This result may be generalized as

$$V = \frac{1}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2}\mathbf{p} \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \tag{38}$$

where $\mathbf{r}$ locates the field point P, and $\mathbf{r}'$ determines the dipole center. Equation (38) is independent of any coordinate system.

The dipole moment **p** will appear again when we discuss dielectric materials. Since it is equal to the product of the charge and the separation, neither the dipole moment nor the potential will change as Q increases and **d** decreases, provided the product remains constant. The limiting case of a *point dipole* is achieved when we let **d** approach zero and Q approach infinity such that the product **p** is finite.

Turning our attention to the resultant fields, it is interesting to note that the potential field is now proportional to the inverse *square* of the distance, and the electric field intensity is proportional to the inverse *cube* of the distance from the dipole. Each field falls off faster than the corresponding field for the point charge, but this is no more than we should expect because the opposite charges appear to be closer together at greater distances and to act more like a single point charge of zero Coulombs.

Symmetrical arrangements of larger numbers of point charges produce fields proportional to the inverse of higher and higher powers of r. These charge distributions are called *multipoles*, and they are used in infinite series to approximate more unwieldy charge configurations.

D4.9. An electric dipole located at the origin in free space has a moment $\mathbf{p} = 3\mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_z$ nC · m. (*a*) Find V at $P_A(2, 3, 4)$. (*b*) Find V at $r = 2.5$, $\theta = 30^\circ$, $\phi = 40^\circ$.

Ans. 0.23 V; 1.97 V

D4.10. A dipole of moment $\mathbf{p} = 6\mathbf{a}_z$ nC · m is located at the origin in free space. (*a*) Find V at $P(r = 4, \theta = 20^\circ, \phi = 0^\circ)$. (*b*) Find **E** at P.

Ans. 3.17 V; $1.58\mathbf{a}_r + 0.29\mathbf{a}_\theta$ V/m

4.8 ENERGY DENSITY IN THE ELECTROSTATIC FIELD

We have introduced the potential concept by considering the work done, or energy expended, in moving a point charge around in an electric field, and now we must tie up the loose ends of that discussion by tracing the energy flow one step further.

Bringing a positive charge from infinity into the field of another positive charge requires work, the work being done by the external source moving the charge. Let us imagine that the external source carries the charge up to a point near the fixed charge and then holds it there. Energy must be conserved, and the energy expended in bringing this charge into position now represents potential energy, for if the external source released its hold on the charge, it would accelerate away from the fixed charge, acquiring kinetic energy of its own and the capability of doing work.

In order to find the potential energy present in a system of charges, we must find the work done by an external source in positioning the charges.

We may start by visualizing an empty universe. Bringing a charge Q_1 from infinity to any position requires no work, for there is no field present.[2] The positioning of Q_2 at a point in the field of Q_1 requires an amount of work given by the product of the charge Q_2 and the potential at that point due to Q_1. We represent this potential as $V_{2,1}$, where the first subscript indicates the location and the second subscript the source. That is, $V_{2,1}$ is the potential at the location of Q_2 due to Q_1. Then

$$\text{Work to position } Q_2 = Q_2 V_{2,1}$$

Similarly, we may express the work required to position each additional charge in the field of all those already present:

$$\text{Work to position } Q_3 = Q_3 V_{3,1} + Q_3 V_{3,2}$$
$$\text{Work to position } Q_4 = Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3}$$

and so forth. The total work is obtained by adding each contribution:

$$\begin{aligned}\text{Total positioning work} &= \text{potential energy of field}\\ &= W_E = Q_2 V_{2,1} + Q_3 V_{3,1} + Q_3 V_{3,2} + Q_4 V_{4,1}\\ &\quad + Q_4 V_{4,2} + Q_4 V_{4,3} + \cdots \end{aligned} \tag{39}$$

Noting the form of a representative term in the preceding equation,

$$Q_3 V_{3,1} = Q_3 \frac{Q_1}{4\pi\epsilon_0 R_{13}} = Q_1 \frac{Q_3}{4\pi\epsilon_0 R_{31}}$$

where R_{13} and R_{31} each represent the scalar distance between Q_1 and Q_3, we see that it might equally well have been written as $Q_1 V_{1,3}$. If each term of the total energy expression is replaced by its equal, we have

$$W_E = Q_1 V_{1,2} + Q_1 V_{1,3} + Q_2 V_{2,3} + Q_1 V_{1,4} + Q_2 V_{2,4} + Q_3 V_{3,4} + \cdots \tag{40}$$

Adding the two energy expressions (39) and (40) gives us a chance to simplify the result a little:

$$\begin{aligned} 2W_E = {} & Q_1(V_{1,2} + V_{1,3} + V_{1,4} + \cdots)\\ & + Q_2(V_{2,1} + V_{2,3} + V_{2,4} + \cdots)\\ & + Q_3(V_{3,1} + V_{3,2} + V_{3,4} + \cdots)\\ & + \cdots \end{aligned}$$

Each sum of potentials in parentheses is the combined potential due to all the charges except for the charge at the point where this combined potential is being found. In other words,

$$V_{1,2} + V_{1,3} + V_{1,4} + \cdots = V_1$$

[2] However, somebody in the workshop at infinity had to do an infinite amount of work to create the point charge in the first place! How much energy is required to bring two half-charges into coincidence to make a unit charge?

V_1 is the potential at the location of Q_1 due to the presence of $Q_2, Q_3, \ldots$. We therefore have

$$W_E = \tfrac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3 + \cdots) = \tfrac{1}{2}\sum_{m=1}^{m=N} Q_m V_m \tag{41}$$

In order to obtain an expression for the energy stored in a region of continuous charge distribution, each charge is replaced by $\rho_v dv$, and the summation becomes an integral,

$$W_E = \tfrac{1}{2}\int_{\text{vol}} \rho_v V \, dv \tag{42}$$

Equations (41) and (42) allow us to find the total potential energy present in a system of point charges or distributed volume charge density. Similar expressions may be easily written in terms of line or surface charge density. Usually we prefer to use Eq. (42) and let it represent all the various types of charge which may have to be considered. This may always be done by considering point charges, line charge density, or surface charge density to be continuous distributions of volume charge density over very small regions. We will illustrate such a procedure with an example shortly.

Before we undertake any interpretation of this result, we should consider a few lines of more difficult vector analysis and obtain an expression equivalent to Eq. (42) but written in terms of **E** and **D**.

We begin by making the expression a little bit longer. Using Maxwell's first equation, replace ρ_v by its equal $\nabla \cdot \mathbf{D}$ and make use of a vector identity which is true for any scalar function V and any vector function **D**,

$$\nabla \cdot (V\mathbf{D}) \equiv V(\nabla \cdot \mathbf{D}) + \mathbf{D} \cdot (\nabla V) \tag{43}$$

This may be proved readily by expansion in rectangular coordinates. We then have, successively,

$$W_E = \tfrac{1}{2}\int_{\text{vol}} \rho_v V dv = \tfrac{1}{2}\int_{\text{vol}} (\nabla \cdot \mathbf{D})V \, dv$$

$$= \tfrac{1}{2}\int_{\text{vol}} [\nabla \cdot (V\mathbf{D}) - \mathbf{D} \cdot (\nabla V)] \, dv$$

Using the divergence theorem from Chapter 3, the first volume integral of the last equation is changed into a closed surface integral, where the closed surface surrounds the volume considered. This volume, first appearing in Eq. (42), must contain *every* charge, and there can then be no charges outside of the volume. We may therefore consider the volume as *infinite* in extent if we wish. We have

$$W_E = \tfrac{1}{2}\oint_S (V\mathbf{D}) \cdot d\mathbf{S} - \tfrac{1}{2}\int_{\text{vol}} \mathbf{D} \cdot (\nabla V) \, dv$$

The surface integral is equal to zero, for over this closed surface surrounding the universe we see that V is approaching zero at least as rapidly as $1/r$ (the charges look like point charges from there), and **D** is approaching zero at least as rapidly as $1/r^2$. The integrand therefore approaches zero at least as rapidly as $1/r^3$, while the differential area of the surface, looking more and more like a portion of a sphere, is increasing only as r^2. Consequently, in the limit as $r \to \infty$, the integrand and the integral both approach zero. Substituting $\mathbf{E} = -\nabla V$ in the remaining volume integral, we have our answer,

$$W_E = \frac{1}{2}\int_{\text{vol}} \mathbf{D}\cdot\mathbf{E}\,dv = \frac{1}{2}\int_{\text{vol}} \epsilon_0 E^2\,dv \tag{44}$$

We may now use this last expression to calculate the energy stored in the electrostatic field of a section of a coaxial cable or capacitor of length L. We found in Section 3.3 that

$$D_\rho = \frac{a\rho_S}{\rho}$$

Hence,

$$\mathbf{E} = \frac{a\rho_S}{\epsilon_0\rho}\mathbf{a}_\rho$$

where ρ_S is the surface charge density on the inner conductor, whose radius is a. Thus,

$$W_E = \frac{1}{2}\int_0^L \int_0^{2\pi} \int_a^b \epsilon_0 \frac{a^2\rho_S^2}{\epsilon_0^2\rho^2}\rho\,d\rho\,d\phi\,dz = \frac{\pi\,L\,a^2\rho_S^2}{\epsilon_0}\ln\frac{b}{a}$$

This same result may be obtained from Eq. (42). We choose the outer conductor as our zero-potential reference, and the potential of the inner cylinder is then

$$V_a = -\int_b^a E_\rho\,d\rho = -\int_b^a \frac{a\rho_S}{\epsilon_0\rho}\,d\rho = \frac{a\rho_S}{\epsilon_0}\ln\frac{b}{a}$$

The surface charge density ρ_S at $\rho = a$ can be interpreted as a volume charge density $\rho_v = \rho_S/t$, extending from $\rho = a - \frac{1}{2}t$ to $\rho = a + \frac{1}{2}t$, where $t \ll a$. The integrand in Eq. (42) is therefore zero everywhere between the cylinders (where the volume charge density is zero), as well as at the outer cylinder (where the potential is zero). The integration is therefore performed only within the thin cylindrical shell at $\rho = a$,

$$W_E = \frac{1}{2}\int_{\text{vol}} \rho_v\,V\,dV = \frac{1}{2}\int_0^L \int_0^{2\pi} \int_{a-t/2}^{a+t/2} \frac{\rho_S}{t}a\frac{\rho_S}{\epsilon_0}\ln\frac{b}{a}\rho\,d\rho\,d\phi\,dz$$

from which

$$W_E = \frac{a^2\rho_S^2\ln(b/a)}{\epsilon_0}\pi L$$

once again.

This expression takes on a more familiar form if we recognize the total charge on the inner conductor as $Q = 2\pi a L\rho_S$. Combining this with the potential difference between the cylinders, V_a, we see that

$$W_E = \frac{1}{2}QV_a$$

which should be familiar as the energy stored in a capacitor.

The question of where the energy is stored in an electric field has not yet been answered. Potential energy can never be pinned down precisely in terms of physical location. Someone lifts a pencil, and the pencil acquires potential energy. Is the energy stored in the molecules of the pencil, in the gravitational field between the pencil and the earth, or in some obscure place? Is the energy in a capacitor stored in the charges themselves, in the field, or where? No one can offer any proof for his or her own private opinion, and the matter of deciding may be left to the philosophers.

Electromagnetic field theory makes it easy to believe that the energy of an electric field or a charge distribution is stored in the field itself, for if we take Eq. (44), an exact and rigorously correct expression,

$$W_E = \frac{1}{2}\int_{\text{vol}} \mathbf{D}\cdot\mathbf{E}\,dv$$

and write it on a differential basis,

$$dW_E = \tfrac{1}{2}\mathbf{D}\cdot\mathbf{E}\,dv$$

or

$$\frac{dW_E}{dv} = \tfrac{1}{2}\mathbf{D}\cdot\mathbf{E} \qquad (45)$$

we obtain a quantity $\frac{1}{2}\mathbf{D}\cdot\mathbf{E}$, which has the dimensions of an energy density, or joules per cubic meter. We know that if we integrate this energy density over the entire field-containing volume, the result is truly the total energy present, but we have no more justification for saying that the energy stored in each differential volume element dv is $\frac{1}{2}\mathbf{D}\cdot\mathbf{E}\,dv$ than we have for looking at Eq. (42) and saying that the stored energy is $\frac{1}{2}\rho_v V\,dv$. The interpretation afforded by Eq. (45), however, is a convenient one, and we will use it until proved wrong.

D4.11. Find the energy stored in free space for the region 2 mm $< r <$ 3 mm, $0 < \theta < 90^\circ$, $0 < \phi < 90^\circ$, given the potential field $V =$: (a) $\dfrac{200}{r}$ V; (b) $\dfrac{300\cos\theta}{r^2}$ V.

Ans. 46.4 μJ; 36.7 J

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1. Attwood, S. S. *Electric and Magnetic Fields*. 3d ed. New York: John Wiley & Sons, 1949. There are a large number of well-drawn field maps of various charge distributions, including the dipole field. Vector analysis is not used.
2. Skilling, H. H. (See Suggested References for Chapter 3.) Gradient is described on pp. 19–21.
3. Thomas, G. B., Jr., and R. L. Finney. (See Suggested References for Chapter 1.) The directional derivative and the gradient are presented on pp. 823–30.

CHAPTER 4 PROBLEMS

WWW Quizzes

4.1 The value of $\mathbf{E}$ at $P(\rho = 2, \phi = 40^\circ, z = 3)$ is given as $\mathbf{E} = 100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z$ V/m. Determine the incremental work required to move a 20 μC charge a distance of 6 μm: (*a*) in the direction of $\mathbf{a}_\rho$; (*b*) in the direction of $\mathbf{a}_\phi$; (*c*) in the direction of $\mathbf{a}_z$; (*d*) in the direction of $\mathbf{E}$; (*e*) in the direction of $\mathbf{G} = 2\mathbf{a}_x - 3\mathbf{a}_y + 4\mathbf{a}_z$.

4.2 A positive point charge of magnitude q_1 lies at the origin. Derive an expression for the incremental work done in moving a second point charge q_2 through a distance dx from the starting position (x, y, z), in the direction of $-\mathbf{a}_x$.

4.3 If $\mathbf{E} = 120\mathbf{a}_\rho$ V/m, find the incremental amount of work done in moving a 50-μC charge a distance of 2 mm from (*a*) $P(1, 2, 3)$ toward $Q(2, 1, 4)$; (*b*) $Q(2, 1, 4)$ toward $P(1, 2, 3)$.

4.4 An electric field in free space is given by $\mathbf{E} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ V/m. Find the work done in moving a 1-μC charge through this field (*a*) from (1, 1, 1) to (0, 0, 0); (*b*) from $(\rho = 2, \phi = 0)$ to $(\rho = 2, \phi = 90^\circ)$; (*c*) from $(r = 10, \theta = \theta_0)$ to $(r = 10, \theta = \theta_0 + 180^\circ)$.

4.5 Compute the value of $\int_A^P \mathbf{G} \cdot d\mathbf{L}$ for $\mathbf{G} = 2y\mathbf{a}_x$ with $A(1, -1, 2)$ and $P(2, 1, 2)$ using the path (*a*) straight-line segments $A(1, -1, 2)$ to $B(1, 1, 2)$ to $P(2, 1, 2)$; (*b*) straight-line segments $A(1, -1, 2)$ to $C(2, -1, 2)$ to $P(2, 1, 2)$.

4.6 An electric field in free space is given as $\mathbf{E} = x\,\hat{\mathbf{a}}_x + 4z\,\hat{\mathbf{a}}_y + 4y\,\hat{\mathbf{a}}_z$. Given $V(1, 1, 1) = 10$ V, determine $V(3, 3, 3)$.

4.7 Let $\mathbf{G} = 3xy^2\mathbf{a}_x + 2z\mathbf{a}_y$ Given an initial point $P(2, 1, 1)$ and a final point $Q(4, 3, 1)$, find $\int \mathbf{G} \cdot d\mathbf{L}$ using the path (*a*) straight line: $y = x - 1$, $z = 1$; (*b*) parabola: $6y = x^2 + 2$, $z = 1$.

4.8 Given $\mathbf{E} = -x\mathbf{a}_x + y\mathbf{a}_y$, (*a*) find the work involved in moving a unit positive charge on a circular arc, the circle centered at the origin, from $x = a$ to $x = y = a/\sqrt{2}$; (*b*) verify that the work done in moving the charge around the full circle from $x = a$ is zero.

4.9 A uniform surface charge density of 20 nC/m^2 is present on the spherical surface $r = 0.6$ cm in free space. (*a*) Find the absolute potential at $P(r = 1 \text{ cm}, \theta = 25^\circ, \phi = 50^\circ)$. (*b*) Find V_{AB}, given points $A(r = 2 \text{ cm}, \theta = 30^\circ, \phi = 60^\circ)$ and $B(r = 3 \text{ cm}, \theta = 45^\circ, \phi = 90^\circ)$.

4.10 A sphere of radius a carries a surface charge density of ρ_{s0} C/m^2. (*a*) Find the absolute potential at the sphere surface. (*b*) A grounded conducting shell of radius b where $b > a$ is now positioned around the charged sphere. What is the potential at the inner sphere surface in this case?

4.11 Let a uniform surface charge density of 5 nC/m^2 be present at the $z = 0$ plane, a uniform line charge density of 8 nC/m be located at $x = 0$, $z = 4$, and a point charge of 2 μC be present at $P(2, 0, 0)$. If $V = 0$ at $M(0, 0, 5)$, find V at $N(1, 2, 3)$.

4.12 In spherical coordinates, $\mathbf{E} = 2r/(r^2 + a^2)^2\mathbf{a}_r$ V/m. Find the potential at any point, using the reference $(a) V = 0$ at infinity; (b) $V = 0$ at $r = 0$; $(c) V = 100$ V at $r = a$.

4.13 Three identical point charges of 4 pC each are located at the corners of an equilateral triangle 0.5 mm on a side in free space. How much work must be done to move one charge to a point equidistant from the other two and on the line joining them?

4.14 Given the electric field $\mathbf{E} = (y + 1)\mathbf{a}_x + (x - 1)\mathbf{a}_y + 2\mathbf{a}_z$ find the potential difference between the points (a) $(2, -2, -1)$ and $(0, 0, 0)$; (b) $(3, 2, -1)$ and $(-2, -3, 4)$.

4.15 Two uniform line charges, 8 nC/m each, are located at $x = 1$, $z = 2$, and at $x = -1$, $y = 2$ in free space. If the potential at the origin is 100 V, find V at $P(4, 1, 3)$.

4.16 A spherically symmetric charge distribution in free space (with $0 < r < \infty$) is known to have a potential function $V(r) = V_0 a^2/r^2$, where V_0 and a are constants. (a) Find the electric field intensity. (b) Find the volume charge density. (c) Find the charge contained inside radius a. (d) Find the total energy stored in the charge (or equivalently, in its electric field).

4.17 Uniform surface charge densities of 6 and 2 nC/m^2 are present at $\rho = 2$ and 6 cm, respectively, in free space. Assume $V = 0$ at $\rho = 4$ cm, and calculate V at (a) $\rho = 5$ cm; (b) $\rho = 7$ cm.

4.18 Find the potential at the origin produced by a line charge $\rho_L = kx/(x^2 + a^2)$ extending along the x axis from $x = a$ to $+\infty$, where $a > 0$. Assume a zero reference at infinity.

4.19 The annular surface 1 cm $< \rho <$ 3 cm, $z = 0$, carries the nonuniform surface charge density $\rho_s = 5\rho$ nC/m^2. Find V at $P(0, 0, 2$ cm$)$ if $V = 0$ at infinity.

4.20 In a certain medium, the electric potential is given by

$$V(x) = \frac{\rho_0}{a\epsilon_0}\left(1 - e^{-ax}\right)$$

where ρ_0 and a are constants. (a) Find the electric field intensity, $\mathbf{E}$. (b) Find the potential difference between the points $x = d$ and $x = 0$. (c) If the medium permittivity is given by $\epsilon(x) = \epsilon_0 e^{ax}$, find the electric flux density, $\mathbf{D}$, and the volume charge density, ρ_v, in the region. (d) Find the stored energy in the region $(0 < x < d)$, $(0 < y < 1)$, $(0 < z < 1)$.

4.21 Let $V = 2xy^2z^3 + 3\ln(x^2 + 2y^2 + 3z^2)$ V in free space. Evaluate each of the following quantities at $P(3, 2, -1)$ (a) V; (b) $|V|$; (c) $\mathbf{E}$; (d) $|\mathbf{E}|$; (e) $\mathbf{a}_N$; (f) $\mathbf{D}$.

4.22 A line charge of infinite length lies along the z axis and carries a uniform linear charge density of ρ_ℓ C/m. A perfectly conducting cylindrical shell, whose axis is the z axis, surrounds the line charge. The cylinder (of radius b), is at ground potential. Under these conditions, the potential function inside the cylinder ($\rho < b$) is given by

$$V(\rho) = k - \frac{\rho_\ell}{2\pi\epsilon_0}\ln(\rho)$$

where k is a constant. (a) Find k in terms of given or known parameters. (b) Find the electric field strength, **E**, for $\rho < b$. (c) Find the electric field strength, **E**, for $\rho > b$. (d) Find the stored energy in the electric field *per unit length* in the z direction within the volume defined by $\rho > a$, where $a < b$.

4.23 It is known that the potential is given as $V = 80\rho^{0.6}$ V. Assuming free space conditions, find. (a) **E**; (b) the volume charge density at $\rho = 0.5$ m; (c) the total charge lying within the closed surface $\rho = 0.6$, $0 < z < 1$.

4.24 A certain spherically symmetric charge configuration in free space produces an electric field given in spherical coordinates by

$$\mathbf{E}(r) = \begin{cases} (\rho_0 r^2)/(100\epsilon_0)\,\mathbf{a}_r \text{ V/m} & (r \le 10) \\ (100\rho_0)/(\epsilon_0 r^2)\,\mathbf{a}_r \text{ V/m} & (r \ge 10) \end{cases}$$

where ρ_0 is a constant. (a) Find the charge density as a function of position. (b) Find the absolute potential as a function of position in the two regions, $r \le 10$ and $r \ge 10$. (c) Check your result of part b by using the gradient. (d) Find the stored energy in the charge by an integral of the form of Eq. (43). (e) Find the stored energy in the field by an integral of the form of Eq. (45).

4.25 Within the cylinder $\rho = 2$, $0 < z < 1$, the potential is given by $V = 100 + 50\rho + 150\rho\sin\phi$V. ($a$) Find V, **E**, **D**, and ρ_v at $P(1, 60^\circ, 0.5)$ in free space. (b) How much charge lies within the cylinder?

4.26 Let us assume that we have a very thin, square, imperfectly conducting plate 2 m on a side, located in the plane $z = 0$ with one corner at the origin such that it lies entirely within the first quadrant. The potential at any point in the plate is given as $V = -e^{-x}\sin y$. (a) An electron enters the plate at $x = 0$, $y = \pi/3$ with zero initial velocity; in what direction is its initial movement? (b) Because of collisions with the particles in the plate, the electron achieves a relatively low velocity and little acceleration (the work that the field does on it is converted largely into heat). The electron therefore moves approximately along a streamline. Where does it leave the plate and in what direction is it moving at the time?

4.27 Two point charges, 1 nC at (0, 0, 0.1) and −1 nC at (0, 0, −0.1), are in free space. (a) Calculate V at $P(0.3, 0, 0.4)$. (b) Calculate $|\mathbf{E}|$ at P. (c) Now treat the two charges as a dipole at the origin and find V at P.

4.28 Use the electric field intensity of the dipole [Section 4.7, Eq. (35)] to find the difference in potential between points at θ_a and θ_b, each point having the same r and ϕ coordinates. Under what conditions does the answer agree with Eq. (33), for the potential at θ_a?

4.29 A dipole having a moment $\mathbf{p} = 3\mathbf{a}_x - 5\mathbf{a}_y + 10\mathbf{a}_z$ nC · m is located at $Q(1, 2, -4)$ in free space. Find V at $P(2, 3, 4)$.

4.30 A dipole for which $\mathbf{p} = 10\epsilon_0\mathbf{a}_z$ C · m is located at the origin. What is the equation of the surface on which $E_z = 0$ but $\mathbf{E} \neq 0$?

4.31 A potential field in free space is expressed as $V = 20/(xyz)$ V. (*a*) Find the total energy stored within the cube $1 < x, y, z < 2$. (*b*) What value would be obtained by assuming a uniform energy density equal to the value at the center of the cube?

4.32 (*a*) Using Eq. (35), find the energy stored in the dipole field in the region $r > a$. (*b*) Why can we not let a approach zero as a limit?

4.33 A copper sphere of radius 4 cm carries a uniformly distributed total charge of 5 μC in free space. (*a*) Use Gauss's law to find **D** external to the sphere. (*b*) Calculate the total energy stored in the electrostatic field. (*c*) Use $W_E = Q^2/(2C)$ to calculate the capacitance of the isolated sphere.

4.34 A sphere of radius a contains volume charge of uniform density ρ_0 C/m^3. Find the total stored energy by applying (*a*) Eq. (42); (*b*) Eq. (44).

4.35 Four 0.8 nC point charges are located in free space at the corners of a square 4 cm on a side. (*a*) Find the total potential energy stored. (*b*) A fifth 0.8 nC charge is installed at the center of the square. Again find the total stored energy.

4.36 Surface charge of uniform density ρ_s lies on a spherical shell of radius b, centered at the origin in free space. (*a*) Find the absolute potential everywhere, with zero reference at infinity. (*b*) Find the stored energy in the sphere by considering the charge density and the potential in a two-dimensional version of Eq. (42). (*c*) Find the stored energy in the electric field and show that the results of parts (*b*) and (*c*) are identical.

CHAPTER 5

Conductors and Dielectrics

In this chapter, we apply the methods we have learned to some of the materials with which an engineer must work. In the first part of the chapter, we consider conducting materials by describing the parameters that relate current to an applied electric field. This leads to a general definition of Ohm's law. We then develop methods of evaluating resistances of conductors in a few simple geometric forms. Conditions that must be met at a conducting boundary are obtained next, and this knowledge leads to a discussion of the method of images. The properties of semiconductors are described to conclude the discussion of conducting media.

In the second part of the chapter, we consider insulating materials, or dielectrics. Such materials differ from conductors in that ideally, there is no free charge that can be transported within them to produce conduction current. Instead, all charge is confined to molecular or lattice sites by coulomb forces. An applied electric field has the effect of displacing the charges slightly, leading to the formation of ensembles of electric dipoles. The extent to which this occurs is measured by the relative permittivity, or dielectric constant. Polarization of the medium may modify the electric field, whose magnitude and direction may differ from the values it would have in a different medium or in free space. Boundary conditions for the fields at interfaces between dielectrics are developed to evaluate these differences.

It should be noted that most materials will possess both dielectric and conductive properties; that is, a material considered a dielectric may be slightly conductive, and a material that is mostly conductive may be slightly polarizable. These departures from the ideal cases lead to some interesting behavior, particularly as to the effects on electromagnetic wave propagation, as we will see later. ■

5.1 CURRENT AND CURRENT DENSITY

Electric charges in motion constitute a *current*. The unit of current is the ampere (A), defined as a rate of movement of charge passing a given reference point (or crossing a given reference plane) of one coulomb per second. Current is symbolized by I, and therefore

$$I = \frac{dQ}{dt} \tag{1}$$

Current is thus defined as the motion of positive charges, even though conduction in metals takes place through the motion of electrons, as we will see shortly.

In field theory, we are usually interested in events occurring at a point rather than within a large region, and we find the concept of *current density,* measured in amperes per square meter (A/m^2), more useful. Current density is a vector[1] represented by $\mathbf{J}$.

The increment of current ΔI crossing an incremental surface ΔS normal to the current density is

$$\Delta I = J_N \Delta S$$

and in the case where the current density is not perpendicular to the surface,

$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{S}$$

Total current is obtained by integrating,

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{2}$$

Current density may be related to the velocity of volume charge density at a point. Consider the element of charge $\Delta Q = \rho_v \Delta v = \rho_v \, \Delta S \, \Delta L$, as shown in Figure 5.1$a$. To simplify the explanation, assume that the charge element is oriented with its edges parallel to the coordinate axes and that it has only an x component of velocity. In the time interval Δt, the element of charge has moved a distance Δx, as indicated in Figure 5.1b. We have therefore moved a charge $\Delta Q = \rho_v \, \Delta S \, \Delta x$ through a reference plane perpendicular to the direction of motion in a time increment Δt, and the resulting current is

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \, \Delta S \frac{\Delta x}{\Delta t}$$

As we take the limit with respect to time, we have

$$\Delta I = \rho_v \, \Delta S \, v_x$$

[1] Current is not a vector, for it is easy to visualize a problem in which a total current I in a conductor of nonuniform cross section (such as a sphere) may have a different direction at each point of a given cross section. Current in an exceedingly fine wire, or a *filamentary current*, is occasionally defined as a vector, but we usually prefer to be consistent and give the direction to the filament, or path, and not to the current.

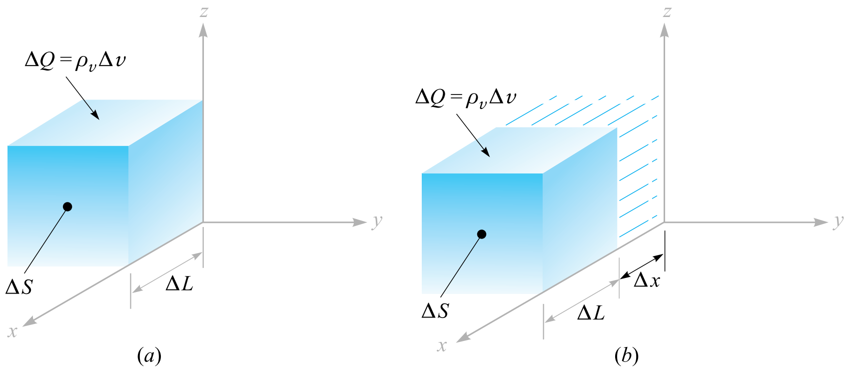


Figure 5.1 An increment of charge, $\Delta Q = \rho_v \Delta S \Delta L$, which moves a distance Δx in a time Δt, produces a component of current density in the limit of $J_x = \rho_v v_x$.

where v_x represents the x component of the velocity $\mathbf{v}$.[2] In terms of current density, we find

$$J_x = \rho_v \, v_x$$

and in general

$$\boxed{\mathbf{J} = \rho_v \mathbf{v}} \tag{3}$$

This last result shows clearly that charge in motion constitutes a current. We call this type of current a *convection current,* and $\mathbf{J}$ or $\rho_v \mathbf{v}$ is the *convection current density*. Note that the convection current density is related linearly to charge density as well as to velocity. The mass rate of flow of cars (cars per square foot per second) in the Holland Tunnel could be increased either by raising the density of cars per cubic foot, or by going to higher speeds, if the drivers were capable of doing so.

D5.1. Given the vector current density $\mathbf{J} = 10\rho^2 z \mathbf{a}_\rho - 4\rho \cos^2\phi \, \mathbf{a}_\phi$ mA/m^2: (a) find the current density at $P(\rho = 3, \phi = 30^\circ, z = 2)$; ($b$) determine the total current flowing outward through the circular band $\rho = 3$, $0 < \phi < 2\pi$, $2 < z < 2.8$.

Ans. $180\mathbf{a}_\rho - 9\mathbf{a}_\phi$ mA/m^2; 3.26 A

5.2 CONTINUITY OF CURRENT

The introduction of the concept of current is logically followed by a discussion of the conservation of charge and the continuity equation. The principle of conservation of charge states simply that charges can be neither created nor destroyed, although equal

[2] The lowercase v is used both for volume and velocity. Note, however, that velocity always appears as a vector $\mathbf{v}$, a component v_x, or a magnitude $|\mathbf{v}|$, whereas volume appears only in differential form as dv or Δv.

amounts of positive and negative charge may be *simultaneously* created, obtained by separation, or lost by recombination.

The continuity equation follows from this principle when we consider any region bounded by a closed surface. The current through the closed surface is

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S}$$

and this *outward flow* of positive charge must be balanced by a decrease of positive charge (or perhaps an increase of negative charge) within the closed surface. If the charge inside the closed surface is denoted by Q_i, then the rate of decrease is $-dQ_i/dt$ and the principle of conservation of charge requires

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{dQ_i}{dt} \tag{4}$$

It might be well to answer here an often-asked question. "Isn't there a sign error? I thought $I = dQ/dt$." The presence or absence of a negative sign depends on what current and charge we consider. In circuit theory we usually associate the current flow *into* one terminal of a capacitor with the time rate of increase of charge on that plate. The current of (4), however, is an *outward-flowing* current.

Equation (4) is the integral form of the continuity equation; the differential, or point, form is obtained by using the divergence theorem to change the surface integral into a volume integral:

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv$$

We next represent the enclosed charge Q_i by the volume integral of the charge density,

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = -\frac{d}{dt}\int_{\text{vol}} \rho_v\, dv$$

If we agree to keep the surface constant, the derivative becomes a partial derivative and may appear within the integral,

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = \int_{\text{vol}} -\frac{\partial \rho_v}{\partial t}\, dv$$

from which we have our point form of the continuity equation,

$$(\nabla \cdot \mathbf{J}) = -\frac{\partial \rho_v}{\partial t} \tag{5}$$

Remembering the physical interpretation of divergence, this equation indicates that the current, or charge per second, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

As a numerical example illustrating some of the concepts from the last two sections, let us consider a current density that is directed radially outward and decreases exponentially with time,

$$\mathbf{J} = \frac{1}{r}e^{-t}\mathbf{a}_r \text{ A/m}^2$$

Selecting an instant of time $t = 1$ s, we may calculate the total outward current at $r = 5$ m:

$$I = J_r S = \left(\tfrac{1}{5}e^{-1}\right)(4\pi 5^2) = 23.1 \text{ A}$$

At the same instant, but for a slightly larger radius, $r = 6$ m, we have

$$I = J_r S = \left(\tfrac{1}{6}e^{-1}\right)\left(4\pi 6^2\right) = 27.7 \text{ A}$$

Thus, the total current is larger at $r = 6$ than it is at $r = 5$.

To see why this happens, we need to look at the volume charge density and the velocity. We use the continuity equation first:

$$-\frac{\partial \rho_v}{\partial t} = \nabla \cdot \mathbf{J} = \nabla \cdot \left(\frac{1}{r}e^{-t}\mathbf{a}_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r}e^{-t}\right) = \frac{1}{r^2}e^{-t}$$

We next seek the volume charge density by integrating with respect to t. Because ρ_v is given by a partial derivative with respect to time, the "constant" of integration may be a function of r:

$$\rho_v = -\int \frac{1}{r^2}e^{-t}\,dt + K(r) = \frac{1}{r^2}e^{-t} + K(r)$$

If we assume that $\rho_v \to 0$ as $t \to \infty$, then $K(r) = 0$, and

$$\rho_v = \frac{1}{r^2}e^{-t} \text{ C/m}^3$$

We may now use $\mathbf{J} = \rho_v \mathbf{v}$ to find the velocity,

$$v_r = \frac{J_r}{\rho_v} = \frac{\frac{1}{r}e^{-t}}{\frac{1}{r^2}e^{-t}} = r \text{ m/s}$$

The velocity is greater at $r = 6$ than it is at $r = 5$, and we see that some (unspecified) force is accelerating the charge density in an outward direction.

In summary, we have a current density that is inversely proportional to r, a charge density that is inversely proportional to r^2, and a velocity and total current that are proportional to r. All quantities vary as e^{-t}.

D5.2. Current density is given in cylindrical coordinates as $\mathbf{J} = -10^6 z^{1.5}\mathbf{a}_z$ A/m^2 in the region $0 \le \rho \le 20\,\mu$m; for $\rho \ge 20\,\mu$m, $\mathbf{J} = 0$. (*a*) Find the total current crossing the surface $z = 0.1$ m in the $\mathbf{a}_z$ direction. (*b*) If the charge velocity is 2×10^6 m/s at $z = 0.1$ m, find ρ_v there. (*c*) If the volume charge density at $z = 0.15$ m is -2000 C/m^3, find the charge velocity there.

Ans. $-39.7\,\mu$A; -15.8 mC/m^3; 29.0 m/s

5.3 METALLIC CONDUCTORS

Physicists describe the behavior of the electrons surrounding the positive atomic nucleus in terms of the total energy of the electron with respect to a zero reference level for an electron at an infinite distance from the nucleus. The total energy is the sum of the kinetic and potential energies, and because energy must be given to an electron to pull it away from the nucleus, the energy of every electron in the atom is a negative quantity. Even though this picture has some limitations, it is convenient to associate these energy values with orbits surrounding the nucleus, the more negative energies corresponding to orbits of smaller radius. According to the quantum theory, only certain discrete energy levels, or energy states, are permissible in a given atom, and an electron must therefore absorb or emit discrete amounts of energy, or quanta, in passing from one level to another. A normal atom at absolute zero temperature has an electron occupying every one of the lower energy shells, starting outward from the nucleus and continuing until the supply of electrons is exhausted.

In a crystalline solid, such as a metal or a diamond, atoms are packed closely together, many more electrons are present, and many more permissible energy levels are available because of the interaction forces between adjacent atoms. We find that the allowed energies of electrons are grouped into broad ranges, or "bands," each band consisting of very numerous, closely spaced, discrete levels. At a temperature of absolute zero, the normal solid also has every level occupied, starting with the lowest and proceeding in order until all the electrons are located. The electrons with the highest (least negative) energy levels, the valence electrons, are located in the *valence band*. If there are permissible higher-energy levels in the valence band, or if the valence band merges smoothly into a *conduction band*, then additional kinetic energy may be given to the valence electrons by an external field, resulting in an electron flow. The solid is called a *metallic conductor*. The filled valence band and the unfilled conduction band for a conductor at absolute zero temperature are suggested by the sketch in Figure 5.2*a*.

If, however, the electron with the greatest energy occupies the top level in the valence band and a gap exists between the valence band and the conduction band, then

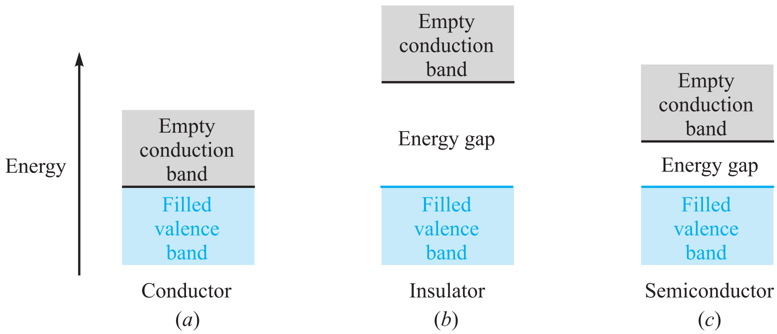


Figure 5.2 The energy-band structure in three different types of materials at 0 K. (*a*) The conductor exhibits no energy gap between the valence and conduction bands. (*b*) The insulator shows a large energy gap. (*c*) The semiconductor has only a small energy gap.

the electron cannot accept additional energy in small amounts, and the material is an insulator. This band structure is indicated in Figure 5.2*b*. Note that if a relatively large amount of energy can be transferred to the electron, it may be sufficiently excited to jump the gap into the next band where conduction can occur easily. Here the insulator breaks down.

An intermediate condition occurs when only a small "forbidden region" separates the two bands, as illustrated by Figure 5.2*c*. Small amounts of energy in the form of heat, light, or an electric field may raise the energy of the electrons at the top of the filled band and provide the basis for conduction. These materials are insulators which display many of the properties of conductors and are called *semiconductors*.

Let us first consider the conductor. Here the valence electrons, or *conduction,* or *free,* electrons, move under the influence of an electric field. With a field $\mathbf{E}$, an electron having a charge $Q = -e$ will experience a force

$$\mathbf{F} = -e\mathbf{E}$$

In free space, the electron would accelerate and continuously increase its velocity (and energy); in the crystalline material, the progress of the electron is impeded by continual collisions with the thermally excited crystalline lattice structure, and a constant average velocity is soon attained. This velocity $\mathbf{v}_d$ is termed the *drift velocity,* and it is linearly related to the electric field intensity by the *mobility* of the electron in the given material. We designate mobility by the symbol μ (mu), so that

$$\mathbf{v}_d = -\mu_e \mathbf{E} \tag{6}$$

where μ_ϵ is the mobility of an electron and is positive by definition. Note that the electron velocity is in a direction opposite to the direction of $\mathbf{E}$. Equation (6) also shows that mobility is measured in the units of square meters per volt-second; typical values[3] are 0.0012 for aluminum, 0.0032 for copper, and 0.0056 for silver.

For these good conductors, a drift velocity of a few centimeters per second is sufficient to produce a noticeable temperature rise and can cause the wire to melt if the heat cannot be quickly removed by thermal conduction or radiation.

Substituting (6) into Eq. (3) of Section 5.1, we obtain

$$\boxed{\mathbf{J} = -\rho_e \mu_e \mathbf{E}} \tag{7}$$

where ρ_e is the free-electron charge density, a negative value. The total charge density ρ_ν is zero because equal positive and negative charges are present in the neutral material. The negative value of ρ_e and the minus sign lead to a current density $\mathbf{J}$ that is in the same direction as the electric field intensity $\mathbf{E}$.

The relationship between $\mathbf{J}$ and $\mathbf{E}$ for a metallic conductor, however, is also specified by the conductivity σ (sigma),

$$\boxed{\mathbf{J} = \sigma \mathbf{E}} \tag{8}$$

[3] Wert and Thomson, p. 238, listed in the References at the end of this chapter.

where σ is measured is siemens[4] per meter (S/m). One siemens (1 S) is the basic unit of conductance in the SI system and is defined as one ampere per volt. Formerly, the unit of conductance was called the mho and was symbolized by an *inverted* Ω. Just as the siemens honors the Siemens brothers, the reciprocal unit of resistance that we call the ohm (1 Ω is one volt per ampere) honors Georg Simon Ohm, a German physicist who first described the current-voltage relationship implied by Eq. (8). We call this equation the *point form of Ohm's law;* we will look at the more common form of Ohm's law shortly.

First, however, it is informative to note the conductivity of several metallic conductors; typical values (in siemens per meter) are 3.82×10^7 for aluminum, 5.80×10^7 for copper, and 6.17×10^7 for silver. Data for other conductors may be found in Appendix C. On seeing data such as these, it is only natural to assume that we are being presented with *constant* values; this is essentially true. Metallic conductors obey Ohm's law quite faithfully, and it is a *linear* relationship; the conductivity is constant over wide ranges of current density and electric field intensity. Ohm's law and the metallic conductors are also described as *isotropic,* or having the same properties in every direction. A material which is not isotropic is called *anisotropic,* and we shall mention such a material in Chapter 6.

The conductivity is a function of temperature, however. The resistivity, which is the reciprocal of the conductivity, varies almost linearly with temperature in the region of room temperature, and for aluminum, copper, and silver it increases about 0.4 percent for a 1-K rise in temperature.[5] For several metals the resistivity drops abruptly to zero at a temperature of a few kelvin; this property is termed *superconductivity*. Copper and silver are not superconductors, although aluminum is (for temperatures below 1.14 K).

If we now combine Equations (7) and (8), conductivity may be expressed in terms of the charge density and the electron mobility,

$$\sigma = -\rho_e \mu_e \tag{9}$$

From the definition of mobility (6), it is now satisfying to note that a higher temperature infers a greater crystalline lattice vibration, more impeded electron progress for a given electric field strength, lower drift velocity, lower mobility, lower conductivity from Eq. (9), and higher resistivity as stated.

The application of Ohm's law in point form to a macroscopic (visible to the naked eye) region leads to a more familiar form. Initially, assume that **J** and **E** are *uniform,* as they are in the cylindrical region shown in Figure 5.3. Because they are uniform,

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} = JS \tag{10}$$

[4] This is the family name of two German-born brothers, Karl Wilhelm and Werner von Siemens, who were famous engineer-inventors in the nineteenth century. Karl became a British subject and was knighted, becoming Sir William Siemens.

[5] Copious temperature data for conducting materials are available in the *Standard Handbook for Electrical Engineers,* listed among the References at the end of this chapter.

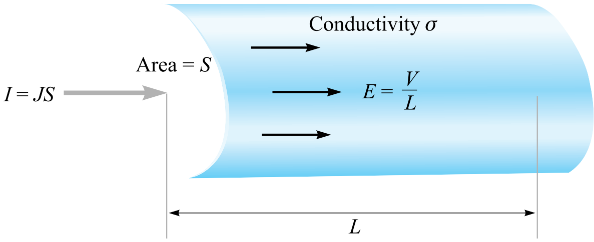


Figure 5.3 Uniform current density J and electric field intensity E in a cylindrical region of length L and cross-sectional area S. Here $V = IR$, where $R = L/\sigma S$.

and

$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{L} = -\mathbf{E} \cdot \int_b^a d\mathbf{L} = -\mathbf{E} \cdot \mathbf{L}_{ba}$$
$$= \mathbf{E} \cdot \mathbf{L}_{ab} \tag{11}$$

or

$$V = EL$$

Thus

$$J = \frac{I}{S} = \sigma E = \sigma \frac{V}{L}$$

or

$$V = \frac{L}{\sigma S} I$$

The ratio of the potential difference between the two ends of the cylinder to the current entering the more positive end, however, is recognized from elementary circuit theory as the *resistance* of the cylinder, and therefore

$$\boxed{V = IR} \tag{12}$$

where

$$\boxed{R = \frac{L}{\sigma S}} \tag{13}$$

Equation (12) is, of course, known as *Ohm's law*, and Eq. (13) enables us to compute the resistance R, measured in ohms (abbreviated as Ω), of conducting objects which possess uniform fields. If the fields are not uniform, the resistance may still be defined as the ratio of V to I, where V is the potential difference between two specified equipotential surfaces in the material and I is the total current crossing the more positive surface into the material. From the general integral relationships in Eqs. (10) and (11), and from Ohm's law (8), we may write this general expression for resistance when the fields are nonuniform,

$$R = \frac{V_{ab}}{I} = \frac{-\int_b^a \mathbf{E} \cdot d\mathbf{L}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{S}} \tag{14}$$

The line integral is taken between two equipotential surfaces in the conductor, and the surface integral is evaluated over the more positive of these two equipotentials. We cannot solve these nonuniform problems at this time, but we should be able to solve several of them after reading Chapter 6.

EXAMPLE 5.1

As an example of the determination of the resistance of a cylinder, we find the resistance of a 1-mile length of #16 copper wire, which has a diameter of 0.0508 in.

Solution. The diameter of the wire is $0.0508 \times 0.0254 = 1.291 \times 10^{-3}$ m, the area of the cross section is $\pi(1.291 \times 10^{-3}/2)^2 = 1.308 \times 10^{-6}$ m^2, and the length is 1609 m. Using a conductivity of 5.80×10^7 S/m, the resistance of the wire is, therefore,

$$R = \frac{1609}{(5.80 \times 10^7)(1.308 \times 10^{-6})} = 21.2\ \Omega$$

This wire can safely carry about 10 A dc, corresponding to a current density of $10/(1.308 \times 10^{-6}) = 7.65 \times 10^6$ A/m^2, or 7.65 A/mm^2. With this current, the potential difference between the two ends of the wire is 212 V, the electric field intensity is 0.312 V/m, the drift velocity is 0.000 422 m/s, or a little more than one furlong a week, and the free-electron charge density is -1.81×10^{10} C/m^3, or about one electron within a cube two angstroms on a side.

D5.3. Find the magnitude of the current density in a sample of silver for which $\sigma = 6.17 \times 10^7$ S/m and $\mu_e = 0.0056$ m^2/V · s if (*a*) the drift velocity is 1.5 μm/s ; (*b*) the electric field intensity is 1 mV/m; (*c*) the sample is a cube 2.5 mm on a side having a voltage of 0.4 mV between opposite faces; (*d*) the sample is a cube 2.5 mm on a side carrying a total current of 0.5 A.

Ans. 16.5 kA/m^2; 61.7 kA/m^2; 9.9 MA/m^2; 80.0 kA/m^2

D5.4. A copper conductor has a diameter of 0.6 in. and it is 1200 ft long. Assume that it carries a total dc current of 50 A. (*a*) Find the total resistance of the conductor. (*b*) What current density exists in it? (*c*) What is the dc voltage between the conductor ends? (*d*) How much power is dissipated in the wire?

Ans. 0.035 Ω; 2.74×10^5 A/m^2; 1.73 V; 86.4 W

5.4 CONDUCTOR PROPERTIES AND BOUNDARY CONDITIONS

Once again, we must temporarily depart from our assumed static conditions and let time vary for a few microseconds to see what happens when the charge distribution is suddenly unbalanced within a conducting material. Suppose, for the sake of argument, that there suddenly appear a number of electrons in the interior of a conductor. The electric fields set up by these electrons are not counteracted by any positive charges, and the electrons therefore begin to accelerate away from each other. This continues until the electrons reach the surface of the conductor or until a number of electrons equal to the number injected have reached the surface.

Here, the outward progress of the electrons is stopped, for the material surrounding the conductor is an insulator not possessing a convenient conduction band. No charge may remain within the conductor. If it did, the resulting electric field would force the charges to the surface.

Hence the final result within a conductor is zero charge density, and a surface charge density resides on the exterior surface. This is one of the two characteristics of a good conductor.

The other characteristic, stated for static conditions in which no current may flow, follows directly from Ohm's law: the electric field intensity within the conductor is zero. Physically, we see that if an electric field were present, the conduction electrons would move and produce a current, thus leading to a nonstatic condition.

Summarizing for electrostatics, no charge and no electric field may exist at any point *within* a conducting material. Charge may, however, appear on the surface as a surface charge density, and our next investigation concerns the fields *external* to the conductor.

We wish to relate these external fields to the charge on the surface of the conductor. The problem is a simple one, and we may first talk our way to the solution with a little mathematics.

If the external electric field intensity is decomposed into two components, one tangential and one normal to the conductor surface, the tangential component is seen to be zero. If it were not zero, a tangential force would be applied to the elements of the surface charge, resulting in their motion and nonstatic conditions. Because static conditions are assumed, the tangential electric field intensity and electric flux density are zero.

Gauss's law answers our questions concerning the normal component. The electric flux leaving a small increment of surface must be equal to the charge residing on that incremental surface. The flux cannot penetrate into the conductor, for the total field there is zero. It must then leave the surface normally. Quantitatively, we may say that the electric flux density in coulombs per square meter leaving the surface normally is equal to the surface charge density in coulombs per square meter, or $D_N = \rho_S$.

If we use some of our previously derived results in making a more careful analysis (and incidentally introducing a general method which we must use later), we should set up a boundary between a conductor and free space (Figure 5.4) showing tangential

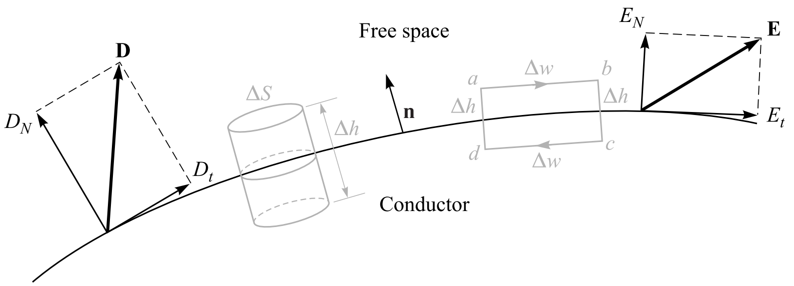


Figure 5.4 An appropriate closed path and gaussian surface are used to determine boundary conditions at a boundary between a conductor and free space; $E_t = 0$ and $D_N = \rho_S$.

and normal components of **D** and **E** on the free-space side of the boundary. Both fields are zero in the conductor. The tangential field may be determined by applying Section 4.5, Eq. (21),

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

around the small closed path *abcda*. The integral must be broken up into four parts

$$\int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$$

Remembering that $\mathbf{E} = 0$ within the conductor, we let the length from a to b or c to d be Δw and from b to c or d to a be Δh, and obtain

$$E_t \Delta w - E_{N,\text{at}\,b} \tfrac{1}{2} \Delta h + E_{N,\text{at}\,a} \tfrac{1}{2} \Delta h = 0$$

As we allow Δh to approach zero, keeping Δw small but finite, it makes no difference whether or not the normal fields are equal at a and b, for Δh causes these products to become negligibly small. Hence, $E_t \Delta w = 0$ and, therefore, $E_t = 0$.

The condition on the normal field is found most readily by considering D_N rather than E_N and choosing a small cylinder as the gaussian surface. Let the height be Δh and the area of the top and bottom faces be ΔS. Again, we let Δh approach zero. Using Gauss's law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

we integrate over the three distinct surfaces

$$\int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{sides}} = Q$$

and find that the last two are zero (for different reasons). Then

$$D_N \Delta S = Q = \rho_S \Delta S$$

or

$$D_N = \rho_S$$

These are the desired *boundary conditions* for the conductor-to-free-space boundary in electrostatics,

$$\boxed{D_t = E_t = 0} \tag{15}$$

$$\boxed{D_N = \epsilon_0 E_N = \rho_S} \tag{16}$$

The electric flux leaves the conductor in a direction normal to the surface, and the value of the electric flux density is numerically equal to the surface charge density. Equations (15) and (16) can be more formally expressed using the vector fields

$$\mathbf{E} \times \mathbf{n}\Big|_s = 0 \tag{17}$$

$$\mathbf{D} \cdot \mathbf{n}\Big|_s = \rho_s \tag{18}$$

where **n** is the unit normal vector at the surface that points *away* from the conductor, as shown in Figure 5.4, and where both operations are evaluated at the conductor surface, s. Taking the cross product or the dot product of either field quantity with **n** gives the tangential or the normal component of the field, respectively.

An immediate and important consequence of a zero tangential electric field intensity is the fact that a conductor surface is an equipotential surface. The evaluation of the potential difference between any two points on the surface by the line integral leads to a zero result, because the path may be chosen on the surface itself where $\mathbf{E} \cdot d\mathbf{L} = 0$.

To summarize the principles which apply to conductors in electrostatic fields, we may state that

1. The static electric field intensity inside a conductor is zero.
2. The static electric field intensity at the surface of a conductor is everywhere directed normal to that surface.
3. The conductor surface is an equipotential surface.

Using these three principles, there are a number of quantities that may be calculated at a conductor boundary, given a knowledge of the potential field.

EXAMPLE 5.2

Given the potential,

$$V = 100(x^2 - y^2)$$

and a point $P(2, -1, 3)$ that is stipulated to lie on a conductor-to-free-space boundary, find V, **E**, **D**, and ρ_S at P, and also the equation of the conductor surface.

Solution. The potential at point P is

$$V_P = 100[2^2 - (-1)^2] = 300 \text{ V}$$

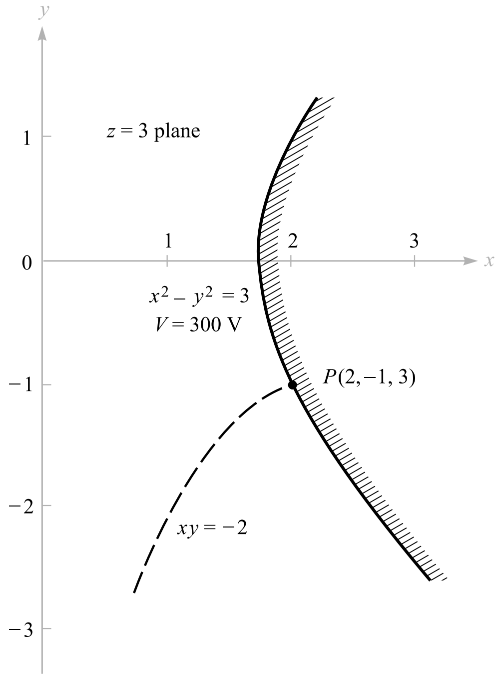


Figure 5.5 Given point $P(2, -1, 3)$ and the potential field, $V = 100(x^2 - y^2)$, we find the equipotential surface through P is $x^2 - y^2 = 3$, and the streamline through P is $xy = -2$.

Because the conductor is an equipotential surface, the potential at the entire surface must be 300 V. Moreover, if the conductor is a solid object, then the potential everywhere in and on the conductor is 300 V, for $\mathbf{E} = 0$ within the conductor.

The equation representing the locus of all points having a potential of 300 V is

$$300 = 100(x^2 - y^2)$$

or

$$x^2 - y^2 = 3$$

This is therefore the equation of the conductor surface; it happens to be a hyperbolic cylinder, as shown in Figure 5.5. Let us assume arbitrarily that the solid conductor lies above and to the right of the equipotential surface at point P, whereas free space is down and to the left.

Next, we find $\mathbf{E}$ by the gradient operation,

$$\mathbf{E} = -100\nabla(x^2 - y^2) = -200x\mathbf{a}_x + 200y\mathbf{a}_y$$

At point P,

$$\mathbf{E}_p = -400\mathbf{a}_x - 200\mathbf{a}_y \text{ V/m}$$

Because $\mathbf{D} = \epsilon_0 \mathbf{E}$, we have

$$\mathbf{D}_P = 8.854 \times 10^{-12} \mathbf{E}_P = -3.54\mathbf{a}_x - 1.771\mathbf{a}_y \ \ \text{nC/m}^2$$

The field is directed downward and to the left at P; it is normal to the equipotential surface. Therefore,

$$D_N = |\mathbf{D}_P| = 3.96 \text{ nC/m}^2$$

Thus, the surface charge density at P is

$$\rho_{S,P} = D_N = 3.96 \text{ nC/m}^2$$

Note that if we had taken the region to the left of the equipotential surface as the conductor, the **E** field would *terminate* on the surface charge and we would let $\rho_S = -3.96 \text{ nC/m}^2$.

EXAMPLE 5.3

Finally, let us determine the equation of the streamline passing through P.

Solution. We see that

$$\frac{E_y}{E_x} = \frac{200y}{-200x} = -\frac{y}{x} = \frac{dy}{dx}$$

Thus,

$$\frac{dy}{y} + \frac{dx}{x} = 0$$

and

$$\ln y + \ln x = C_1$$

Therefore,

$$xy = C_2$$

The line (or surface) through P is obtained when $C_2 = (2)(-1) = -2$. Thus, the streamline is the trace of another hyperbolic cylinder,

$$xy = -2$$

This is also shown on Figure 5.5.

D5.5. Given the potential field in free space, $V = 100 \sinh 5x \sin 5y \ V$, and a point $P(0.1, 0.2, 0.3)$, find at P: (*a*) V; (*b*) $\mathbf{E}$; (*c*) $|\mathbf{E}|$; (*d*) $|\rho_S|$ if it is known that P lies on a conductor surface.

Ans. 43.8 V; $-474\mathbf{a}_x - 140.8\mathbf{a}_y$ V/m; 495 V/m; 4.38 nC/m^2

5.5 THE METHOD OF IMAGES

One important characteristic of the dipole field that we developed in Chapter 4 is the infinite plane at zero potential that exists midway between the two charges. Such a plane may be represented by a vanishingly thin conducting plane that is infinite in extent. The conductor is an equipotential surface at a potential $V = 0$, and the electric field intensity is therefore normal to the surface. Thus, if we replace the dipole configuration shown in Figure 5.6*a* with the single charge and conducting plane shown in Figure 5.6*b*, the fields in the upper half of each figure are the same. Below the conducting plane, all fields are zero, as we have not provided any charges in that region. Of course, we might also substitute a single negative charge below a conducting plane for the dipole arrangement and obtain equivalence for the fields in the lower half of each region.

If we approach this equivalence from the opposite point of view, we begin with a single charge above a perfectly conducting plane and then see that we may maintain the same fields above the plane by removing the plane and locating a negative charge at a symmetrical location below the plane. This charge is called the *image* of the original charge, and it is the negative of that value.

If we can do this once, linearity allows us to do it again and again, and thus *any* charge configuration above an infinite ground plane may be replaced by an arrangement composed of the given charge configuration, its image, and no conducting plane. This is suggested by the two illustrations of Figure 5.7. In many cases, the potential field of the new system is much easier to find since it does not contain the conducting plane with its unknown surface charge distribution.

As an example of the use of images, let us find the surface charge density at $P(2, 5, 0)$ on the conducting plane $z = 0$ if there is a line charge of 30 nC/m located at $x = 0$, $z = 3$, as shown in Figure 5.8*a*. We remove the plane and install an image line charge of -30 nC/m at $x = 0$, $z = -3$, as illustrated in Figure 5.8*b*. The field at P may now be obtained by superposition of the known fields of the line

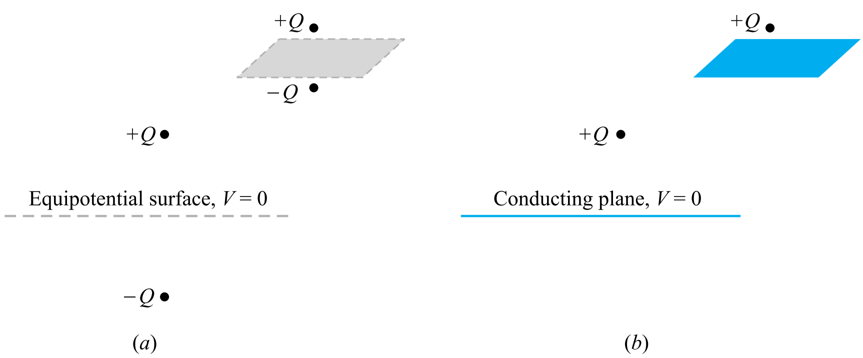


Figure 5.6 (*a*) Two equal but opposite charges may be replaced by (*b*) a single charge and a conducting plane without affecting the fields above the $V = 0$ surface.

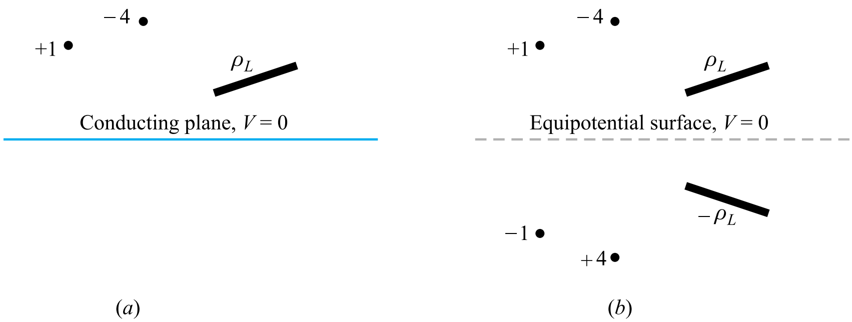


Figure 5.7 (*a*) A given charge configuration above an infinite conducting plane may be replaced by (*b*) the given charge configuration plus the image configuration, without the conducting plane.

charges. The radial vector from the positive line charge to P is $\mathbf{R}_+ = 2\mathbf{a}_x - 3\mathbf{a}_z$, while $\mathbf{R}_- = 2\mathbf{a}_x + 3\mathbf{a}_z$. Thus, the individual fields are

$$\mathbf{E}_+ = \frac{\rho_L}{2\pi\epsilon_0 R_+}\mathbf{a}_{R+} = \frac{30 \times 10^{-9}}{2\pi\epsilon_0\sqrt{13}}\,\frac{2\mathbf{a}_x - 3\mathbf{a}_z}{\sqrt{13}}$$

and

$$\mathbf{E}_- = \frac{30 \times 10^{-9}}{2\pi\epsilon_0\sqrt{13}}\,\frac{2\mathbf{a}_x + 3\mathbf{a}_z}{\sqrt{13}}$$

Adding these results, we have

$$\mathbf{E} = \frac{-180 \times 10^{-9}\mathbf{a}_z}{2\pi\epsilon_0(13)} = -249\mathbf{a}_z \text{ V/m}$$

This then is the field at (or just above) P in both the configurations of Figure 5.8, and it is certainly satisfying to note that the field is normal to the conducting plane, as it must be. Thus, $\mathbf{D} = \epsilon_0\mathbf{E} = -2.20\mathbf{a}_z$ nC/m^2, and because this is directed *toward* the conducting plane, ρ_S is negative and has a value of -2.20 nC/m^2 at P.

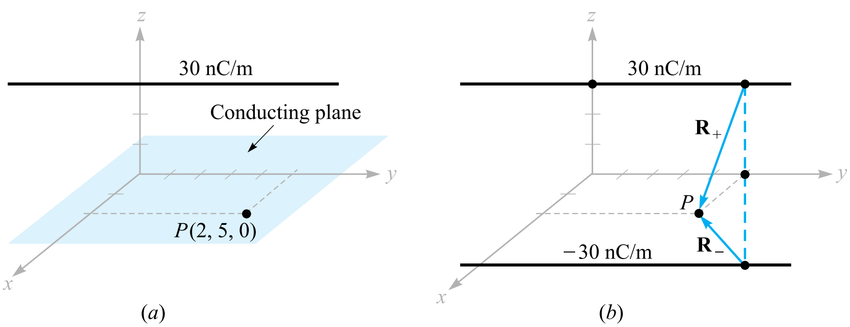


Figure 5.8 (*a*) A line charge above a conducting plane. (*b*) The conductor is removed, and the image of the line charge is added.

D5.6. A perfectly conducting plane is located in free space at $x = 4$, and a uniform infinite line charge of 40 nC/m lies along the line $x = 6$, $y = 3$. Let $V = 0$ at the conducting plane. At $P(7, -1, 5)$ find: (*a*) V; (*b*) $\mathbf{E}$.

Ans. 317 V; $-45.3\mathbf{a}_x - 99.2\mathbf{a}_y$ V/m

5.6 SEMICONDUCTORS

If we now turn our attention to an intrinsic semiconductor material, such as pure germanium or silicon, two types of current carriers are present, electrons and holes. The electrons are those from the top of the filled valence band that have received sufficient energy (usually thermal) to cross the relatively small forbidden band into the conduction band. The forbidden-band energy gap in typical semiconductors is of the order of one electronvolt. The vacancies left by these electrons represent unfilled energy states in the valence band which may also move from atom to atom in the crystal. The vacancy is called a hole, and many semiconductor properties may be described by treating the hole as if it had a positive charge of e, a mobility, μ_h, and an effective mass comparable to that of the electron. Both carriers move in an electric field, and they move in opposite directions; hence each contributes a component of the total current which is in the same direction as that provided by the other. The conductivity is therefore a function of both hole and electron concentrations and mobilities,

$$\sigma = -\rho_e \mu_e + \rho_h \mu_h \tag{19}$$

For pure, or *intrinsic,* silicon, the electron and hole mobilities are 0.12 and 0.025, respectively, whereas for germanium, the mobilities are, respectively, 0.36 and 0.17. These values are given in square meters per volt-second and range from 10 to 100 times as large as those for aluminum, copper, silver, and other metallic conductors.[6] These mobilities are given for a temperature of 300 K.

The electron and hole concentrations depend strongly on temperature. At 300 K the electron and hole volume charge densities are both 0.0024 C/m^{3}in magnitude in intrinsic silicon and 3.0 C/m^3 in intrinsic germanium. These values lead to conductivities of 0.000 35 S/m in silicon and 1.6 S/m in germanium. As temperature increases, the mobilities decrease, but the charge densities increase very rapidly. As a result, the conductivity of silicon increases by a factor of 10 as the temperature increases from 300 to about 330 K and decreases by a factor of 10 as the temperature drops from 300 to about 275 K. Note that the conductivity of the intrinsic semiconductor increases with temperature, whereas that of a metallic conductor decreases with temperature; this is one of the characteristic differences between the metallic conductors and the intrinsic semiconductors.

[6] Mobility values for semiconductors are given in References 2, 3, and 5 listed at the end of this chapter.

Intrinsic semiconductors also satisfy the point form of Ohm's law; that is, the conductivity is reasonably constant with current density and with the direction of the current density.

The number of charge carriers and the conductivity may both be increased dramatically by adding very small amounts of impurities. *Donor* materials provide additional electrons and form *n-type* semiconductors, whereas *acceptors* furnish extra holes and form *p-type* materials. The process is known as *doping,* and a donor concentration in silicon as low as one part in 10^7 causes an increase in conductivity by a factor of 10^5.

The range of value of the conductivity is extreme as we go from the best insulating materials to semiconductors and the finest conductors. In siemens per meter, σ ranges from 10^{-17} for fused quartz, 10^{-7} for poor plastic insulators, and roughly unity for semiconductors to almost 10^8 for metallic conductors at room temperature. These values cover the remarkably large range of some 25 orders of magnitude.

D5.7. Using the values given in this section for the electron and hole mobilities in silicon at 300 K, and assuming hole and electron charge densities are 0.0029 C/m^3 and -0.0029 C/m^3, respectively, find: (*a*) the component of the conductivity due to holes; (*b*) the component of the conductivity due to electrons; (*c*) the conductivity.

Ans. 72.5 μS/m; 348 μS/m; 421 μS/m

Quizzes

5.7 THE NATURE OF DIELECTRIC MATERIALS

A dielectric in an electric field can be viewed as a free-space arrangement of microscopic electric dipoles, each of which is composed of a positive and a negative charge whose centers do not quite coincide.These are not free charges, and they cannot contribute to the conduction process. Rather, they are bound in place by atomic and molecular forces and can only shift positions slightly in response to external fields. They are called *bound* charges, in contrast to the free charges that determine conductivity. The bound charges can be treated as any other sources of the electrostatic field. Therefore, we would not need to introduce the dielectric constant as a new parameter or to deal with permittivities different from the permittivity of free space; however, the alternative would be to consider *every charge within a piece of dielectric material*. This is too great a price to pay for using all our previous equations in an unmodified form, and we shall therefore spend some time theorizing about dielectrics in a qualitative way; introducing polarization **P**, permittivity ϵ, and relative permittivity ϵ_r; and developing some quantitative relationships involving these new parameters.

Interactives

The characteristic that all dielectric materials have in common, whether they are solid, liquid, or gas, and whether or not they are crystalline in nature, is their ability to store electric energy. This storage takes place by means of a shift in the relative positions of the internal, bound positive and negative charges against the normal molecular and atomic forces.

This displacement against a restraining force is analogous to lifting a weight or stretching a spring and represents potential energy. The source of the energy is the external field, the motion of the shifting charges resulting perhaps in a transient current through a battery that is producing the field.

The actual mechanism of the charge displacement differs in the various dielectric materials. Some molecules, termed *polar* molecules, have a permanent displacement existing between the centers of "gravity" of the positive and negative charges, and each pair of charges acts as a dipole. Normally the dipoles are oriented in a random way throughout the interior of the material, and the action of the external field is to align these molecules, to some extent, in the same direction. A sufficiently strong field may even produce an additional displacement between the positive and negative charges.

A *nonpolar* molecule does not have this dipole arrangement until after a field is applied. The negative and positive charges shift in opposite directions against their mutual attraction and produce a dipole that is aligned with the electric field.

Either type of dipole may be described by its dipole moment $\mathbf{p}$, as developed in Section 4.7, Eq. (36),

$$\mathbf{p} = Q\mathbf{d} \tag{20}$$

where Q is the positive one of the two bound charges composing the dipole, and $\mathbf{d}$ is the vector from the negative to the positive charge. We note again that the units of $\mathbf{p}$ are coulomb-meters.

If there are n dipoles per unit volume and we deal with a volume Δv, then there are $n\,\Delta v$ dipoles, and the total dipole moment is obtained by the vector sum,

$$\mathbf{p}_{\text{total}} = \sum_{i=1}^{n\,\Delta v} \mathbf{p}_i$$

If the dipoles are aligned in the same general direction, $\mathbf{p}_{\text{total}}$ may have a significant value. However, a random orientation may cause $\mathbf{p}_{\text{total}}$ to be essentially zero.

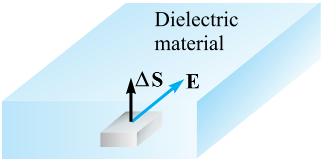
We now define the polarization $\mathbf{P}$ as the *dipole moment per unit volume,*

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\,\Delta v} \mathbf{p}_i \tag{21}$$

with units of coulombs per square meter. We will treat $\mathbf{P}$ as a typical continuous field, even though it is obvious that it is essentially undefined at points within an atom or molecule. Instead, we should think of its value at any point as an average value taken over a sample volume Δv—large enough to contain many molecules ($n\,\Delta v$ in number), but yet sufficiently small to be considered incremental in concept.

Our immediate goal is to show that the bound-volume charge density acts like the free-volume charge density in producing an external field; we will obtain a result similar to Gauss's law.

To be specific, assume that we have a dielectric containing nonpolar molecules. No molecule has a dipole moment, and $\mathbf{P} = 0$ throughout the material. Somewhere in the interior of the dielectric we select an incremental surface element $\Delta\mathbf{S}$, as shown in Figure 5.9*a*, and apply an electric field $\mathbf{E}$. The electric field produces a moment



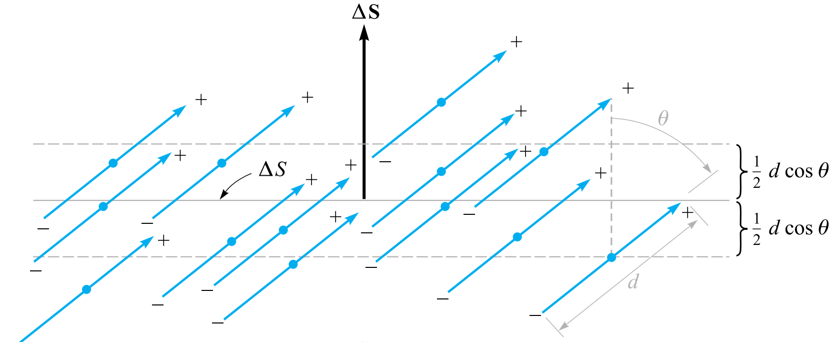


Figure 5.9 (*a*) An incremental surface element ΔS is shown in the interior of a dielectric in which an electric field **E** is present. (*b*) The nonpolar molecules form dipole moments **p** and a polarization **P**. There is a net transfer of bound charge across ΔS.

$\mathbf{p} = Q\mathbf{d}$ in each molecule, such that **p** and **d** make an angle θ with $\Delta\mathbf{S}$, as indicated in Figure 5.9*b*.

The bound charges will now move across $\Delta\mathbf{S}$. Each of the charges associated with the creation of a dipole must have moved a distance $\frac{1}{2}d\cos\theta$ in the direction perpendicular to $\Delta\mathbf{S}$. Thus, any positive charges initially lying below the surface $\Delta\mathbf{S}$ and within the distance $\frac{1}{2}d\cos\theta$ of the surface must have crossed $\Delta\mathbf{S}$ going upward. Also, any negative charges initially lying above the surface and within that distance $(\frac{1}{2}d\cos\theta)$ from $\Delta\mathbf{S}$ must have crossed $\Delta\mathbf{S}$ going downward. Therefore, because there are n molecules/m^3, the net total charge that crosses the elemental surface in an upward direction is equal to $nQd\cos\theta\,\Delta S$, or

$$\Delta Q_b = nQ\mathbf{d}\cdot\Delta\mathbf{S}$$

where the subscript on Q_b reminds us that we are dealing with a bound charge and not a free charge. In terms of the polarization, we have

$$\Delta Q_b = \mathbf{P}\cdot\Delta\mathbf{S}$$

If we interpret $\Delta\mathbf{S}$ as an element of a *closed* surface inside the dielectric material, then the direction of $\Delta\mathbf{S}$ is outward, and the net increase in the bound charge *within* the closed surface is obtained through the integral

$$Q_b = -\oint_S \mathbf{P}\cdot d\mathbf{S} \tag{22}$$

This last relationship has some resemblance to Gauss's law, and we may now generalize our definition of electric flux density so that it applies to media other than free space. We first write Gauss's law in terms of $\epsilon_0\mathbf{E}$ and Q_T, the *total* enclosed charge, bound plus free:

$$Q_T = \oint_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S} \tag{23}$$

where

$$Q_T = Q_b + Q$$

and Q is the total *free* charge enclosed by the surface S. Note that the free charge appears without a subscript because it is the most important type of charge and will appear in Maxwell's equations.

Combining these last three equations, we obtain an expression for the free charge enclosed,

$$Q = Q_T - Q_b = \oint_S (\epsilon_0 \mathbf{E} + \mathbf{P}) \cdot d\mathbf{S} \tag{24}$$

D is now defined in more general terms than was done in Chapter 3,

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}} \tag{25}$$

There is thus an added term to **D** that appears when polarizable material is present. Thus,

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S} \tag{26}$$

where Q is the free charge enclosed.

Utilizing the several volume charge densities, we have

$$Q_b = \int_v \rho_b \, dv$$

$$Q = \int_v \rho_v \, dv$$

$$Q_T = \int_v \rho_T \, dv$$

With the help of the divergence theorem, we may therefore transform Eqs. (22), (23), and (26) into the equivalent divergence relationships,

$$\nabla \cdot \mathbf{P} = -\rho_b$$

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_T$$

$$\boxed{\nabla \cdot \mathbf{D} = \rho_v} \tag{27}$$

We will emphasize only Eq. (26) and (27), the two expressions involving the free charge, in the work that follows.

In order to make any real use of these new concepts, it is necessary to know the relationship between the electric field intensity **E** and the polarization **P** that results. This relationship will, of course, be a function of the type of material, and we will essentially limit our discussion to those isotropic materials for which **E** and **P** are linearly related. In an isotropic material, the vectors **E** and **P** are always parallel, regardless of the orientation of the field. Although most engineering dielectrics are linear for moderate-to-large field strengths and are also isotropic, single crystals may be anisotropic. The periodic nature of crystalline materials causes dipole moments to be formed most easily along the crystal axes, and not necessarily in the direction of the applied field.

In *ferroelectric* materials, the relationship between **P** and **E** not only is nonlinear, but also shows hysteresis effects; that is, the polarization produced by a given electric field intensity depends on the past history of the sample. Important examples of this type of dielectric are barium titanate, often used in ceramic capacitors, and Rochelle salt.

The linear relationship between **P** and **E** is

$$\boxed{\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}} \tag{28}$$

where χ_e (chi) is a dimensionless quantity called the *electric susceptibility* of the material.

Using this relationship in Eq. (25), we have

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \chi_e \epsilon_0 \mathbf{E} = (\chi_e + 1)\epsilon_0 \mathbf{E}$$

The expression within the parentheses is now defined as

$$\epsilon_r = \chi_e + 1 \tag{29}$$

This is another dimensionless quantity, and it is known as the *relative permittivity*, or *dielectric constant* of the material. Thus,

$$\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E} \tag{30}$$

where

$$\boxed{\epsilon = \epsilon_0 \epsilon_r} \tag{31}$$

and ϵ is the *permittivity*. The dielectric constants are given for some representative materials in Appendix C.

Anisotropic dielectric materials cannot be described in terms of a simple susceptibility or permittivity parameter. Instead, we find that each component of **D** may be a function of every component of **E**, and $\mathbf{D} = \epsilon \mathbf{E}$ becomes a matrix equation where **D** and **E** are each 3×1 column matrices and ϵ is a 3×3 square matrix. Expanding the matrix equation gives

$$D_x = \epsilon_{xx} E_x + \epsilon_{xy} E_y + \epsilon_{xz} E_z$$
$$D_y = \epsilon_{yx} E_x + \epsilon_{yy} E_y + \epsilon_{yz} E_z$$
$$D_z = \epsilon_{zx} E_x + \epsilon_{zy} E_y + \epsilon_{zz} E_z$$

Note that the elements of the matrix depend on the selection of the coordinate axes in the anisotropic material. Certain choices of axis directions lead to simpler matrices.[7]

D and **E** (and **P**) are no longer parallel, and although $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ remains a valid equation for anisotropic materials, we may continue to use $\mathbf{D} = \epsilon\mathbf{E}$ only by interpreting it as a matrix equation. We will concentrate our attention on linear isotropic materials and reserve the general case for a more advanced text.

In summary, then, we now have a relationship between **D** and **E** that depends on the dielectric material present,

$$\mathbf{D} = \epsilon\mathbf{E} \tag{30}$$

where

$$\epsilon = \epsilon_0\epsilon_r \tag{31}$$

This electric flux density is still related to the free charge by either the point or integral form of Gauss's law:

$$\nabla \cdot \mathbf{D} = \rho_v \tag{27}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \tag{26}$$

Use of the relative permittivity, as indicated by Eq. (31), makes consideration of the polarization, dipole moments, and bound charge unnecessary. However, when anisotropic or nonlinear materials must be considered, the relative permittivity, in the simple scalar form that we have discussed, is no longer applicable.

EXAMPLE 5.4

We locate a slab of Teflon in the region $0 \leq x \leq a$, and assume free space where $x < 0$ and $x > a$. Outside the Teflon there is a uniform field $\mathbf{E}_{\text{out}} = E_0\mathbf{a}_x$ V/m. We seek values for **D**, **E**, and **P** everywhere.

Solution. The dielectric constant of the Teflon is 2.1, and thus the electric susceptibility is 1.1.

Outside the slab, we have immediately $\mathbf{D}_{\text{out}} = \epsilon_0 E_0\mathbf{a}_x$. Also, as there is no dielectric material there, $\mathbf{P}_{\text{out}} = 0$. Now, any of the last four or five equations will enable us to relate the several fields inside the material to each other. Thus

$$\mathbf{D}_{\text{in}} = 2.1\epsilon_0\mathbf{E}_{\text{in}} \qquad (0 \leq x \leq a)$$
$$\mathbf{P}_{\text{in}} = 1.1\epsilon_0\mathbf{E}_{\text{in}} \qquad (0 \leq x \leq a)$$

[7] A more complete discussion of this matrix may be found in the Ramo, Whinnery, and Van Duzer reference listed at the end of this chapter.

As soon as we establish a value for any of these three fields within the dielectric, the other two can be found immediately. The difficulty lies in crossing over the boundary from the known fields external to the dielectric to the unknown ones within it. To do this we need a boundary condition, and this is the subject of the next section. We will complete this example then.

In the remainder of this text we will describe polarizable materials in terms of **D** and ϵ rather than **P** and χ_e. We will limit our discussion to isotropic materials.

D5.8. A slab of dielectric material has a relative dielectric constant of 3.8 and contains a uniform electric flux density of 8 nC/m^2. If the material is lossless, find: (*a*) E; (*b*) P; (*c*) the average number of dipoles per cubic meter if the average dipole moment is 10^{-29}C · m.

Ans. 238 V/m; 5.89 nC/m^2; 5.89 × 10^{20} m^{-3}

5.8 BOUNDARY CONDITIONS FOR PERFECT DIELECTRIC MATERIALS

How do we attack a problem in which there are two different dielectrics, or a dielectric and a conductor? This is another example of a *boundary condition*, such as the condition at the surface of a conductor whereby the tangential fields are zero and the normal electric flux density is equal to the surface charge density on the conductor. Now we take the first step in solving a two-dielectric problem, or a dielectric-conductor problem, by determining the behavior of the fields at the dielectric interface.

Let us first consider the interface between two dielectrics having permittivities ϵ_1 and ϵ_2 and occupying regions 1 and 2, as shown in Figure 5.10. We first examine

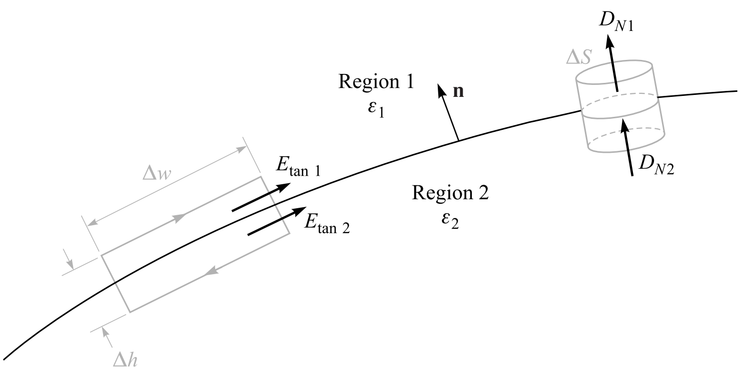


Figure 5.10 The boundary between perfect dielectrics of permittivities ϵ_1 and ϵ_2. The continuity of D_N is shown by the gaussian surface on the right, and the continuity of $E_{\tan}$ is shown by the line integral about the closed path at the left.

the tangential components by using

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

around the small closed path on the left, obtaining

$$E_{\tan 1}\, \Delta w - E_{\tan 2}\, \Delta w = 0$$

The small contribution to the line integral by the normal component of **E** along the sections of length Δh becomes negligible as Δh decreases and the closed path crowds the surface. Immediately, then,

$$\boxed{E_{\tan 1} = E_{\tan 2}} \tag{32}$$

Evidently, Kirchhoff's voltage law is still applicable to this case. Certainly we have shown that the potential difference between any two points on the boundary that are separated by a distance Δw is the same immediately above or below the boundary.

If the tangential electric field intensity is continuous across the boundary, then tangential **D** is discontinuous, for

$$\frac{D_{\tan 1}}{\epsilon_1} = E_{\tan 1} = E_{\tan 2} = \frac{D_{\tan 2}}{\epsilon_2}$$

or

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\epsilon_1}{\epsilon_2} \tag{33}$$

The boundary conditions on the normal components are found by applying Gauss's law to the small "pillbox" shown at the right in Figure 5.10. The sides are again very short, and the flux leaving the top and bottom surfaces is the difference

$$D_{N1}\Delta S - D_{N2}\Delta S = \Delta Q = \rho_S \Delta S$$

from which

$$\boxed{D_{N1} - D_{N2} = \rho_S} \tag{34}$$

What is this surface charge density? It cannot be a *bound* surface charge density, because we are taking the polarization of the dielectric into effect by using a dielectric constant different from unity; that is, instead of considering bound charges in free space, we are using an increased permittivity. Also, it is extremely unlikely that any *free* charge is on the interface, for no free charge is available in the perfect dielectrics we are considering. This charge must then have been placed there deliberately, thus unbalancing the total charge in and on this dielectric body. Except for this special case, then, we may assume ρ_S is zero on the interface and

$$\boxed{D_{N1} = D_{N2}} \tag{35}$$

or the normal component of **D** is continuous. It follows that

$$\epsilon_1 E_{N1} = \epsilon_2 E_{N2} \tag{36}$$

and normal **E** is discontinuous.

Equations (32) and (34) can be written in terms of field vectors in any direction, along with the unit normal to the surface as shown in Figure 5.10. Formally stated, the boundary conditions for the electric flux density and the electric field strength at the surface of a perfect dielectric are

$$(\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{n} = \rho_s \tag{37}$$

which is the general statement of Eq. (32), and

$$(\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{n} = 0 \tag{38}$$

generally states Eq. (34). This construction was used previously in Eqs. (17) and (18) for a conducting surface, in which the normal or tangential components of the fields are obtained through the dot product or cross product with the normal, respectively.

These conditions may be used to show the change in the vectors **D** and **E** at the surface. Let $\mathbf{D}_1$ (and $\mathbf{E}_1$) make an angle θ_1 with a normal to the surface (Figure 5.11). Because the normal components of **D** are continuous,

$$D_{N1} = D_1 \cos\theta_1 = D_2 \cos\theta_2 = D_{N2} \tag{39}$$

The ratio of the tangential components is given by (33) as

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

or

$$\epsilon_2 D_1 \sin\theta_1 = \epsilon_1 D_2 \sin\theta_2 \tag{40}$$

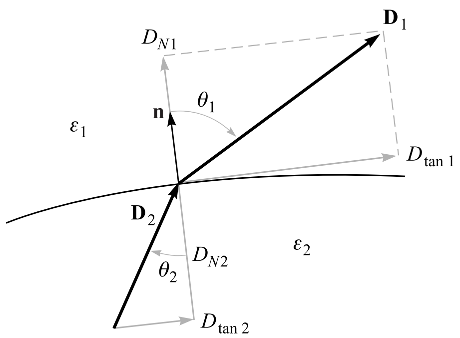


Figure 5.11 The refraction of **D** at a dielectric interface. For the case shown, $\epsilon_1 > \epsilon_2$; $\mathbf{E}_1$ and $\mathbf{E}_2$ are directed along $\mathbf{D}_1$ and $\mathbf{D}_2$, with $D_1 > D_2$ and $E_1 < E_2$.

and the division of this equation by (39) gives

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2} \tag{41}$$

In Figure 5.11 we have assumed that $\epsilon_1 > \epsilon_2$, and therefore $\theta_1 > \theta_2$.

The direction of **E** on each side of the boundary is identical with the direction of **D**, because $\mathbf{D} = \epsilon\mathbf{E}$.

The magnitude of **D** in region 2 may be found from Eq. (39) and (40),

$$D_2 = D_1\sqrt{\cos^2\theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2\theta_1} \tag{42}$$

and the magnitude of $\mathbf{E}_2$ is

$$E_2 = E_1\sqrt{\sin^2\theta_1 + \left(\frac{\epsilon_1}{\epsilon_2}\right)^2 \cos^2\theta_1} \tag{43}$$

An inspection of these equations shows that D is larger in the region of larger permittivity (unless $\theta_1 = \theta_2 = 0^\circ$ where the magnitude is unchanged) and that E is larger in the region of smaller permittivity (unless $\theta_1 = \theta_2 = 90^\circ$, where its magnitude is unchanged).

EXAMPLE 5.5

Complete Example 5.4 by finding the fields within the Teflon ($\epsilon_r = 2.1$), given the uniform external field $\mathbf{E}_{out} = E_0\mathbf{a}_x$ in free space.

Solution. We recall that we had a slab of Teflon extending from $x = 0$ to $x = a$, as shown in Figure 5.12, with free space on both sides of it and an external field $\mathbf{E}_{out} = E_0\mathbf{a}_x$. We also have $\mathbf{D}_{out} = \epsilon_0 E_0\mathbf{a}_x$ and $\mathbf{P}_{out} = 0$.

Inside, the continuity of D_N at the boundary allows us to find that $\mathbf{D}_{in} = \mathbf{D}_{out} = \epsilon_0 E_0\mathbf{a}_x$. This gives us $\mathbf{E}_{in} = \mathbf{D}_{in}/\epsilon = \epsilon_0 E_0\mathbf{a}_x/(\epsilon_r\epsilon_0) = 0.476E_0\mathbf{a}_x$. To get the polarization field in the dielectric, we use $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ and obtain

$$\mathbf{P}_{in} = \mathbf{D}_{in} - \epsilon_0\mathbf{E}_{in} = \epsilon_0 E_0\mathbf{a}_x - 0.476\epsilon_0 E_0\mathbf{a}_x = 0.524\epsilon_0 E_0\mathbf{a}_x$$

Summarizing then gives

$$\mathbf{D}_{in} = \epsilon_0 E_0\mathbf{a}_x \qquad (0 \le x \le a)$$

$$\mathbf{E}_{in} = 0.476E_0\mathbf{a}_x \qquad (0 \le x \le a)$$

$$\mathbf{P}_{in} = 0.524\epsilon_0 E_0\mathbf{a}_x \qquad (0 \le x \le a)$$

A practical problem most often does not provide us with a direct knowledge of the field on either side of the boundary. The boundary conditions must be used to help us determine the fields on both sides of the boundary from the other information that is given.

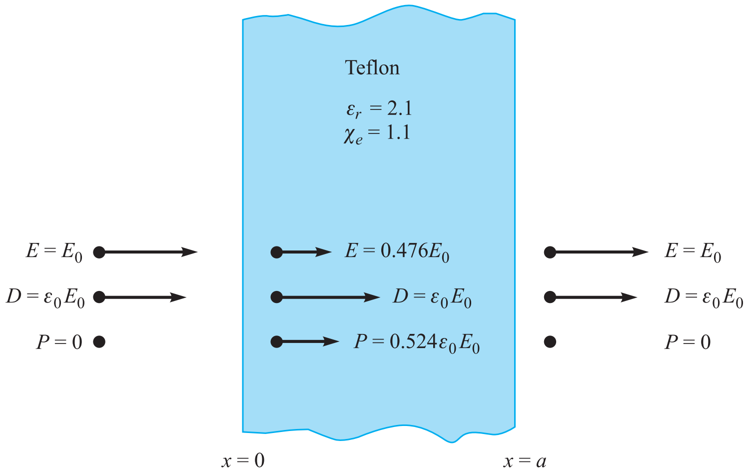


Figure 5.12 A knowledge of the electric field external to the dielectric enables us to find the remaining external fields first and then to use the continuity of normal **D** to begin finding the internal fields.

D5.9. Let Region 1 ($z < 0$) be composed of a uniform dielectric material for which $\epsilon_r = 3.2$, while Region 2 ($z > 0$) is characterized by $\epsilon_r = 2$. Let $\mathbf{D}_1 = -30\mathbf{a}_x + 50\mathbf{a}_y + 70\mathbf{a}_z$ nC/m^2 and find: (*a*) D_{N1}; (*b*) $\mathbf{D}_{t1}$; (*c*) D_{t1}; (*d*) D_1; (*e*) θ_1; (*f*) $\mathbf{P}_1$.

Ans. 70 nC/m^2; $-30\mathbf{a}_x + 50\mathbf{a}_y$ nC/m^2; 58.3 nC/m^2; 91.1 nC/m^2; 39.8°; $-20.6\mathbf{a}_x + 34.4\mathbf{a}_y + 48.1\mathbf{a}_z$ nC/m^2

D5.10. Continue Problem D5.9 by finding: (*a*) $\mathbf{D}_{N2}$; (*b*) $\mathbf{D}_{t2}$; (*c*) $\mathbf{D}_2$; (*d*) $\mathbf{P}_2$; (*e*) θ_2.

Ans. $70\mathbf{a}_z$ nC/m^2; $-18.75\mathbf{a}_x + 31.25\mathbf{a}_y$ nC/m^2; $-18.75\mathbf{a}_x + 31.25\mathbf{a}_y + 70\mathbf{a}_z$ nC/m^2; $-9.38\mathbf{a}_x + 15.63\mathbf{a}_y + 35\mathbf{a}_z$ nC/m^2; 27.5°

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CHAPTER 5 PROBLEMS

5.1 Given the current density $\mathbf{J} = -10^4[\sin(2x)e^{-2y}\mathbf{a}_x + \cos(2x)e^{-2y}\mathbf{a}_y]$ kA/m^2 (*a*) Find the total current crossing the plane $y = 1$ in the $\mathbf{a}_y$ direction in the region $0 < x < 1, 0 < z < 2$. (*b*) Find the total current leaving the region $0 < x, y < 1, 2 < z < 3$ by integrating $\mathbf{J} \cdot d\mathbf{S}$ over the surface of the cube. (*c*) Repeat part (*b*), but use the divergence theorem.

5.2 Given $\mathbf{J} = -10^{-4}(y\mathbf{a}_x + x\mathbf{a}_y)$ A/m^2, find the current crossing the $y = 0$ plane in the $-\mathbf{a}_y$ direction between $z = 0$ and 1, and $x = 0$ and 2.

5.3 Let $\mathbf{J} = 400 \sin\theta/(r^2 + 4)\,\mathbf{a}_r$ A/m^2. (*a*) Find the total current flowing through that portion of the spherical surface $r = 0.8$, bounded by $0.1\pi < \theta < 0.3\pi, 0 < \phi < 2\pi$. (*b*) Find the average value of $\mathbf{J}$ over the defined area.

5.4 If volume charge density is given as $\rho_v = (\cos\omega t)/r^2$ C/m^2 in spherical coordinates, find $\mathbf{J}$. It is reasonable to assume that $\mathbf{J}$ is not a function of θ or ϕ.

5.5 Let $\mathbf{J} = 25/\rho\mathbf{a}_\rho - 20/(\rho^2 + 0.01)\,\mathbf{a}_z$ A/m^2. (*a*) Find the total current crossing the plane $z = 0.2$ in the $\mathbf{a}_z$ direction for $\rho < 0.4$. (*b*) Calculate $\partial\rho_v/\partial t$. (*c*) Find the outward current crossing the closed surface defined by $\rho = 0.01$, $\rho = 0.4$, $z = 0$, and $z = 0.2$. (*d*) Show that the divergence theorem is satisified for $\mathbf{J}$ and the surface specified in part (*c*).

5.6 In spherical coordinates, a current density $\mathbf{J} = -k/(r\sin\theta)\,\mathbf{a}_\theta$ A/m^2 exists in a conducting medium, where k is a constant. Determine the total current in the $\mathbf{a}_z$ direction that crosses a circular disk of radius R, centered on the z axis and located at (*a*) $z = 0$; (*b*) $z = h$.

5.7 Assuming that there is no transformation of mass to energy or vice versa, it is possible to write a continuity equation for mass. (*a*) If we use the continuity equation for charge as our model, what quantities correspond to $\mathbf{J}$ and ρ_v? (*b*) Given a cube 1 cm on a side, experimental data show that the rates at which mass is leaving each of the six faces are 10.25, −9.85, 1.75, −2.00, −4.05, and 4.45 mg/s. If we assume that the cube is an incremental volume element, determine an approximate value for the time rate of change of density at its center.

5.8 A truncated cone has a height of 16 cm. The circular faces on the top and bottom have radii of 2 mm and 0.1 mm, respectively. If the material from which this solid cone is constructed has a conductivity of 2×10^6 S/m, use some good approximations to determine the resistance between the two circular faces.

5.9 (*a*) Using data tabulated in Appendix C, calculate the required diameter for a 2-m-long nichrome wire that will dissipate an average power of 450 W when 120 V rms at 60 Hz is applied to it. (*b*) Calculate the rms current density in the wire.

5.10 A large brass washer has a 2-cm inside diameter, a 5-cm outside diameter, and is 0.5 cm thick. Its conductivity is $\sigma = 1.5 \times 10^7$ S/m. The washer is cut in half along a diameter, and a voltage is applied between the two rectangular faces of one part. The resultant electric field in the interior of the half-washer is $\mathbf{E} = (0.5/\rho)\,\mathbf{a}_\phi$ V/m in cylindrical coordinates, where the z axis is the axis of the washer. (*a*) What potential difference exists between the two rectangular faces? (*b*) What total current is flowing? (*c*) What is the resistance between the two faces?

5.11 Two perfectly conducting cylindrical surfaces of length ℓ are located at $\rho = 3$ and $\rho = 5$ cm. The total current passing radially outward through the medium between the cylinders is 3 A dc. (*a*) Find the voltage and resistance between the cylinders, and **E** in the region between the cylinders, if a conducting material having $\sigma = 0.05$ S/m is present for $3 < \rho < 5$ cm. (*b*) Show that integrating the power dissipated per unit volume over the volume gives the total dissipated power.

5.12 Two identical conducting plates, each having area A, are located at $z = 0$ and $z = d$. The region between plates is filled with a material having z-dependent conductivity, $\sigma(z) = \sigma_0 e^{-z/d}$, where σ_0 is a constant. Voltage V_0 is applied to the plate at $z = d$; the plate at $z = 0$ is at zero potential. Find, in terms of the given parameters, (*a*) the resistance of the material; (*b*) the total current flowing between plates; (*c*) the electric field intensity **E** within the material.

5.13 A hollow cylindrical tube with a rectangular cross section has external dimensions of 0.5 in. by 1 in. and a wall thickness of 0.05 in. Assume that the material is brass, for which $\sigma = 1.5 \times 10^7$ S/m. A current of 200 A dc is flowing down the tube. (*a*) What voltage drop is present across a 1 m length of the tube? (*b*) Find the voltage drop if the interior of the tube is filled with a conducting material for which $\sigma = 1.5 \times 10^5$ S/m.

5.14 A rectangular conducting plate lies in the xy plane, occupying the region $0 < x < a$, $0 < y < b$. An identical conducting plate is positioned directly above and parallel to the first, at $z = d$. The region between plates is filled with material having conductivity $\sigma(x) = \sigma_0 e^{-x/a}$, where σ_0 is a constant. Voltage V_0 is applied to the plate at $z = d$; the plate at $z = 0$ is at zero potential. Find, in terms of the given parameters, (*a*) the electric field intensity **E** within the material; (*b*) the total current flowing between plates; (*c*) the resistance of the material.

5.15 Let $V = 10(\rho + 1)z^2 \cos\phi$ V in free space. (*a*) Let the equipotential surface $V = 20$ V define a conductor surface. Find the equation of the conductor surface. (*b*) Find ρ and $\mathbf{E}$ at that point on the conductor surface where $\phi = 0.2\pi$ and $z = 1.5$. (*c*) Find $|\rho_S|$ at that point.

5.16 A coaxial transmission line has inner and outer conductor radii a and b. Between conductors ($a < \rho < b$) lies a conductive medium whose conductivity is $\sigma(\rho) = \sigma_0/\rho$, where σ_0 is a constant. The inner conductor is charged to potential V_0, and the outer conductor is grounded. (*a*) Assuming dc radial current I per unit length in z, determine the radial current density field $\mathbf{J}$ in A/m^2. (*b*) Determine the electric field intensity $\mathbf{E}$ in terms of I and other parameters, given or known. (*c*) By taking an appropriate line integral of $\mathbf{E}$ as found in part (*b*), find an expression that relates V_0 to I. (*d*) Find an expression for the conductance of the line per unit length, G.

5.17 Given the potential field $V = 100xz/(x^2 + 4)$ V in free space: (*a*) Find $\mathbf{D}$ at the surface $z = 0$. (*b*) Show that the $z = 0$ surface is an equipotential surface. (*c*) Assume that the $z = 0$ surface is a conductor and find the total charge on that portion of the conductor defined by $0 < x < 2$, $-3 < y < 0$.

5.18 Two parallel circular plates of radius a are located at $z = 0$ and $z = d$. The top plate ($z = d$) is raised to potential V_0; the bottom plate is grounded. Between the plates is a conducting material having radial-dependent conductivity, $\sigma(\rho) = \sigma_0\rho$, where σ_0 is a constant. (*a*) Find the ρ-*independent* electric field strength, $\mathbf{E}$, between plates. (*b*) Find the current density, $\mathbf{J}$ between plates. (*c*) Find the total current, I, in the structure. (*d*) Find the resistance between plates.

5.19 Let $V = 20x^2yz - 10z^2$ V in free space. (*a*) Determine the equations of the equipotential surfaces on which $V = 0$ and 60 V. (*b*) Assume these are conducting surfaces and find the surface charge density at that point on the $V = 60$ V surface where $x = 2$ and $z = 1$. It is known that $0 \le V \le 60$ V is the field-containing region. (*c*) Give the unit vector at this point that is normal to the conducting surface and directed toward the $V = 0$ surface.

5.20 Two point charges of $-100\pi\,\mu$C are located at $(2, -1, 0)$ and $(2, 1, 0)$. The surface $x = 0$ is a conducting plane. (*a*) Determine the surface charge density at the origin. (*b*) Determine ρ_S at $P(0, h, 0)$.

5.21 Let the surface $y = 0$ be a perfect conductor in free space. Two uniform infinite line charges of 30 nC/m each are located at $x = 0$, $y = 1$, and $x = 0$, $y = 2$. (*a*) Let $V = 0$ at the plane $y = 0$, and find V at $P(1, 2, 0)$. (*b*) Find $\mathbf{E}$ at P.

5.22 The line segment $x = 0$, $-1 \le y \le 1$, $z = 1$, carries a linear charge density $\rho_L = \pi|y|\,\mu$C/m. Let $z = 0$ be a conducting plane and determine the surface charge density at: (*a*) $(0, 0, 0)$; (*b*) $(0, 1, 0)$.

5.23 A dipole with $\mathbf{p} = 0.1\mathbf{a}_z\ \mu\text{C}\cdot\text{m}$ is located at $A(1, 0, 0)$ in free space, and the $x = 0$ plane is perfectly conducting. (*a*) Find V at $P(2, 0, 1)$. (*b*) Find the equation of the 200 V equipotential surface in rectangular coordinates.

5.24 At a certain temperature, the electron and hole mobilities in intrinsic germanium are given as 0.43 and 0.21 $\text{m}^2/\text{V}\cdot\text{s}$, respectively. If the electron and hole concentrations are both $2.3 \times 10^{19}\ \text{m}^{-3}$, find the conductivity at this temperature.

5.25 Electron and hole concentrations increase with temperature. For pure silicon, suitable expressions are $\rho_h = -\rho_e = 6200T^{1.5}e^{-7000/T}\ \text{C/m}^3$. The functional dependence of the mobilities on temperature is given by $\mu_h = 2.3 \times 10^5 T^{-2.7}\ \text{m}^2/\text{V}\cdot\text{s}$ and $\mu_e = 2.1 \times 10^5 T^{-2.5}\ \text{m}^2/\text{V}\cdot\text{s}$, where the temperature, T, is in degrees Kelvin. Find σ at: (*a*) 0°C; (*b*) 40°C; (*c*) 80°C.

5.26 A semiconductor sample has a rectangular cross section 1.5 by 2.0 mm, and a length of 11.0 mm. The material has electron and hole densities of 1.8×10^{18} and $3.0 \times 10^{15}\ \text{m}^{-3}$, respectively. If $\mu_e = 0.082\ \text{m}^2/\text{V}\cdot\text{s}$ and $\mu_h = 0.0021\ \text{m}^2/\ \text{V}\cdot\text{s}$, find the resistance offered between the end faces of the sample.

5.27 Atomic hydrogen contains 5.5×10^{25} atoms/m^{3}at a certain temperature and pressure. When an electric field of 4 kV/m is applied, each dipole formed by the electron and positive nucleus has an effective length of 7.1×10^{-19} m. (*a*) Find P. (*b*) Find ϵ_r.

5.28 Find the dielectric constant of a material in which the electric flux density is four times the polarization.

5.29 A coaxial conductor has radii $a = 0.8$ mm and $b = 3$ mm and a polystyrene dielectric for which $\epsilon_r = 2.56$. If $\mathbf{P} = (2/\rho)\mathbf{a}_\rho\ \text{nC/m}^2$ in the dielectric, find (*a*) $\mathbf{D}$ and $\mathbf{E}$ as functions of ρ; (*b*) V_{ab} and χ_e. (*c*) If there are 4×10^{19} molecules per cubic meter in the dielectric, find $\mathbf{p}(\rho)$.

5.30 Consider a composite material made up of two species, having number densities N_1 and N_2 molecules/m^3, respectively. The two materials are uniformly mixed, yielding a total number density of $N = N_1 + N_2$. The presence of an electric field $\mathbf{E}$ induces molecular dipole moments $\mathbf{p}_1$ and $\mathbf{p}_2$ within the individual species, whether mixed or not. Show that the dielectric constant of the composite material is given by $\epsilon_r = f\epsilon_{r1} + (1 - f)\epsilon_{r2}$, where f is the number fraction of species 1 dipoles in the composite, and where ϵ_{r1} and ϵ_{r2} are the dielectric constants that the unmixed species would have if each had number density N.

5.31 The surface $x = 0$ separates two perfect dielectrics. For $x > 0$, let $\epsilon_r = \epsilon_{r1} = 3$, while $\epsilon_{r2} = 5$ where $x < 0$. If $\mathbf{E}_1 = 80\mathbf{a}_x - 60\mathbf{a}_y - 30\mathbf{a}_z$ V/m, find (*a*) E_{N1}; (*b*) $\mathbf{E}_{T1}$; (*c*) E_1; (*d*) the angle θ_1 between $\mathbf{E}_1$ and a normal to the surface; (*e*) D_{N2}; (*f*) D_{T2}; (*g*) $\mathbf{D}_2$; (*h*) $\mathbf{P}_2$; (*i*) the angle θ_2 between $\mathbf{E}_2$ and a normal to the surface.

5.32 Two equal but opposite-sign point charges of 3 μC are held x meters apart by a spring that provides a repulsive force given by $F_{sp} = 12(0.5 - x)$ N. Without any force of attraction, the spring would be fully extended to 0.5 m. (*a*) Determine the charge separation. (*b*) What is the dipole moment?

5.33 Two perfect dielectrics have relative permittivities $\epsilon_{r1} = 2$ and $\epsilon_{r2} = 8$. The planar interface between them is the surface $x - y + 2z = 5$. The origin lies in region 1. If $\mathbf{E}_1 = 100\mathbf{a}_x + 200\mathbf{a}_y - 50\mathbf{a}_z$ V/m, find $\mathbf{E}_2$.

5.34 Region 1 ($x \geq 0$) is a dielectric with $\epsilon_{r1} = 2$, while region 2($x < 0$) has $\epsilon_{r2} = 5$. Let $\mathbf{E}_1 = 20\mathbf{a}_x - 10\mathbf{a}_y + 50\mathbf{a}_z$ V/m. (*a*) Find $\mathbf{D}_2$. (*b*) Find the energy density in both regions.

5.35 Let the cylindrical surfaces $\rho = 4$ cm and $\rho = 9$ cm enclose two wedges of perfect dielectrics, $\epsilon_{r1} = 2$ for $0 < \phi < \pi/2$ and $\epsilon_{r2} = 5$ for $\pi/2 < \phi < 2\pi$. If $\mathbf{E}_1 = (2000/\rho)\mathbf{a}_\rho$ V/m, find (*a*) $\mathbf{E}_2$; (*b*) the total electrostatic energy stored in a 1 m length of each region.

CHAPTER 6

Capacitance

Capacitance measures the capability of energy storage in electrical devices. It can be deliberately designed for a specific purpose, or it may exist as an unavoidable by-product of the device structure that one must live with. Understanding capacitance and its impact on device or system operation is critical in every aspect of electrical engineering.

A capacitor is a device that stores energy; energy thus stored can either be associated with accumulated charge or it can be related to the stored electric field, as was discussed in Section 4.8. In fact, one can think of a capacitor as a device that stores electric *flux*, in a similar way that an inductor — an analogous device — stores magnetic flux (or ultimately magnetic field energy). We will explore this in Chapter 8. A primary goal in this chapter is to present the methods for calculating capacitance for a number of cases, including transmission line geometries, and to be able to make judgments on how capacitance will be altered by changes in materials or their configuration. ■

6.1 CAPACITANCE DEFINED

Consider two conductors embedded in a homogeneous dielectric (Figure 6.1). Conductor M_2 carries a total positive charge Q, and M_1 carries an equal negative charge. There are no other charges present, and the *total* charge of the system is zero.

We now know that the charge is carried on the surface as a surface charge density and also that the electric field is normal to the conductor surface. Each conductor is, moreover, an equipotential surface. Because M_2 carries the positive charge, the electric flux is directed from M_2 to M_1, and M_2 is at the more positive potential. In other words, work must be done to carry a positive charge from M_1 to M_2.

Let us designate the potential difference between M_2 and M_1 as V_0. We may now define the *capacitance* of this two-conductor system as the ratio of the magnitude

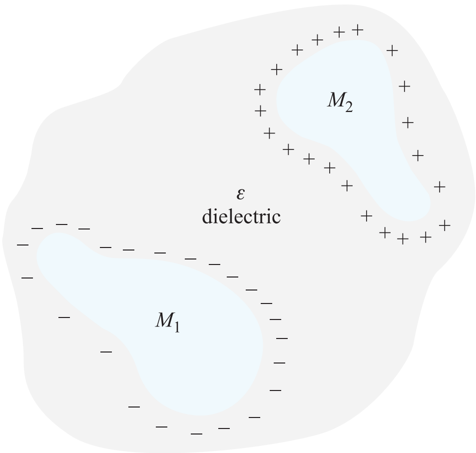

Figure 6.1 Two oppositely charged conductors M_1 and M_2 surrounded by a uniform dielectric. The ratio of the magnitude of the charge on either conductor to the magnitude of the potential difference between them is the capacitance C.

of the total charge on either conductor to the magnitude of the potential difference between conductors,

$$C = \frac{Q}{V_0} \tag{1}$$

In general terms, we determine Q by a surface integral over the positive conductors, and we find V_0 by carrying a unit positive charge from the negative to the positive surface,

$$C = \frac{\oint_S \epsilon \mathbf{E} \cdot d\mathbf{S}}{-\int_-^+ \mathbf{E} \cdot d\mathbf{L}} \tag{2}$$

The capacitance is independent of the potential and total charge, for their ratio is constant. If the charge density is increased by a factor of N, Gauss's law indicates that the electric flux density or electric field intensity also increases by N, as does the potential difference. The capacitance is a function only of the physical dimensions of the system of conductors and of the permittivity of the homogeneous dielectric.

Capacitance is measured in *farads* (F), where a farad is defined as one coulomb per volt. Common values of capacitance are apt to be very small fractions of a farad, and consequently more practical units are the microfarad (μF), the nanofarad (nF), and the picofarad (pF).

6.2 PARALLEL-PLATE CAPACITOR

We can apply the definition of capacitance to a simple two-conductor system in which the conductors are identical, infinite parallel planes with separation d (Figure 6.2). Choosing the lower conducting plane at $z = 0$ and the upper one at $z = d$, a uniform sheet of surface charge $\pm\rho_S$ on each conductor leads to the uniform field [Section 2.5, Eq. (18)]

$$\mathbf{E} = \frac{\rho_S}{\epsilon}\mathbf{a}_z$$

where the permittivity of the homogeneous dielectric is ϵ, and

$$\mathbf{D} = \rho_S \mathbf{a}_z$$

Note that this result could be obtained by applying the boundary condition at a conducting surface (Eq. (18), Chapter 5) at either *one* of the plate surfaces. Referring to the surfaces and their unit normal vectors in Fig. 6.2, where $\mathbf{n}_\ell = \mathbf{a}_z$ and $\mathbf{n}_u = -\mathbf{a}_z$, we find on the lower plane:

$$\mathbf{D} \cdot \mathbf{n}_\ell \Big|_{z=0} = \mathbf{D} \cdot \mathbf{a}_z = \rho_s \;\;\Rightarrow\;\; \mathbf{D} = \rho_s\, \mathbf{a}_z$$

On the upper plane, we get the same result

$$\mathbf{D} \cdot \mathbf{n}_u \Big|_{z=d} = \mathbf{D} \cdot (-\mathbf{a}_z) = -\rho_s \;\;\Rightarrow\;\; \mathbf{D} = \rho_s\, \mathbf{a}_z$$

This is a key advantage of the conductor boundary condition, in that we need to apply it only to a single boundary to obtain the *total* field there (arising from all other sources).

The potential difference between lower and upper planes is

$$V_0 = -\int_{\text{upper}}^{\text{lower}} \mathbf{E} \cdot d\mathbf{L} = -\int_d^0 \frac{\rho_S}{\epsilon} dz = \frac{\rho_S}{\epsilon} d$$

Since the total charge on either plane is infinite, the capacitance is infinite. A more practical answer is obtained by considering planes, each of area S, whose linear dimensions are much greater than their separation d. The electric field and charge

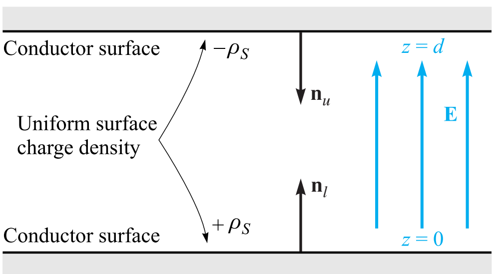

Figure 6.2 The problem of the parallel-plate capacitor. The capacitance per square meter of surface area is ϵ/d.

distribution are then almost uniform at all points not adjacent to the edges, and this latter region contributes only a small percentage of the total capacitance, allowing us to write the familiar result

$$Q = \rho_S S$$

$$V_0 = \frac{\rho_S}{\epsilon} d$$

$$C = \frac{Q}{V_0} = \frac{\epsilon S}{d} \tag{3}$$

More rigorously, we might consider Eq. (3) as the capacitance of a portion of the infinite-plane arrangement having a surface area S. Methods of calculating the effect of the unknown and nonuniform distribution near the edges must wait until we are able to solve more complicated potential problems.

EXAMPLE 6.1

Calculate the capacitance of a parallel-plate capacitor having a mica dielectric, $\epsilon_r = 6$, a plate area of 10 in.^{2}, and a separation of 0.01 in.

Solution. We may find that

$$S = 10 \times 0.0254^2 = 6.45 \times 10^{-3} \text{ m}^2$$

$$d = 0.01 \times 0.0254 = 2.54 \times 10^{-4} \text{ m}$$

and therefore

$$C = \frac{6 \times 8.854 \times 10^{-12} \times 6.45 \times 10^{-3}}{2.54 \times 10^{-4}} = 1.349 \text{ nF}$$

A large plate area is obtained in capacitors of small physical dimensions by stacking smaller plates in 50- or 100-decker sandwiches, or by rolling up foil plates separated by a flexible dielectric.

Table C.1 in Appendix C also indicates that materials are available having dielectric constants greater than 1000.

Finally, the total energy stored in the capacitor is

$$W_E = \tfrac{1}{2}\int_{\text{vol}} \epsilon E^2\, dv = \tfrac{1}{2}\int_0^S \int_0^d \frac{\epsilon\rho_S^2}{\epsilon^2} dz\, dS = \tfrac{1}{2}\frac{\rho_S^2}{\epsilon} Sd = \tfrac{1}{2}\frac{\epsilon S}{d}\frac{\rho_S^2 d^2}{\epsilon^2}$$

or

$$W_E = \tfrac{1}{2} C V_0^2 = \tfrac{1}{2} Q V_0 = \tfrac{1}{2}\frac{Q^2}{C} \tag{4}$$

which are all familiar expressions. Equation (4) also indicates that the energy stored in a capacitor with a fixed potential difference across it increases as the dielectric constant of the medium increases.

D6.1. Find the relative permittivity of the dielectric material present in a parallel-plate capacitor if: (*a*) $S = 0.12\,\text{m}^2$, $d = 80\,\mu\text{m}$, $V_0 = 12$ V, and the capacitor contains $1\,\mu$J of energy; (*b*) the stored energy density is 100 J/m^3, $V_0 = 200$ V, and $d = 45\,\mu$m; (*c*) $E = 200$ kV/m and $\rho_S = 20\,\mu\text{C/m}^2$.

Ans. 1.05; 1.14; 11.3

6.3 SEVERAL CAPACITANCE EXAMPLES

As a first brief example, we choose a coaxial cable or coaxial capacitor of inner radius a, outer radius b, and length L. No great derivational struggle is required, because the potential difference is given as Eq. (11) in Section 4.3, and we find the capacitance very simply by dividing this by the total charge $\rho_L L$ in the length L. Thus,

$$C = \frac{2\pi\epsilon L}{\ln(b/a)} \tag{5}$$

Next we consider a spherical capacitor formed of two concentric spherical conducting shells of radius a and b, $b > a$. The expression for the electric field was obtained previously by Gauss's law,

$$E_r = \frac{Q}{4\pi\epsilon r^2}$$

where the region between the spheres is a dielectric with permittivity ϵ. The expression for potential difference was found from this by the line integral [Section 4.3, Eq. (12)]. Thus,

$$V_{ab} = \frac{Q}{4\pi\epsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Here Q represents the total charge on the inner sphere, and the capacitance becomes

$$C = \frac{Q}{V_{ab}} = \frac{4\pi\epsilon}{\dfrac{1}{a} - \dfrac{1}{b}} \tag{6}$$

If we allow the outer sphere to become infinitely large, we obtain the capacitance of an isolated spherical conductor,

$$C = 4\pi\epsilon a \tag{7}$$

For a diameter of 1 cm, or a sphere about the size of a marble,

$$C = 0.556 \text{ pF}$$

in free space.

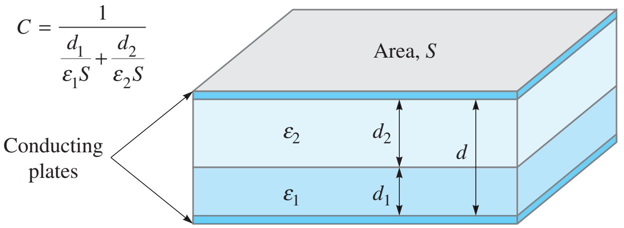

Figure 6.3 A parallel-plate capacitor containing two dielectrics with the dielectric interface parallel to the conducting plates.

Coating this sphere with a different dielectric layer, for which $\epsilon = \epsilon_1$, extending from $r = a$ to $r = r_1$,

$$D_r = \frac{Q}{4\pi r^2}$$

$$E_r = \frac{Q}{4\pi\epsilon_1 r^2} \qquad (a < r < r_1)$$

$$= \frac{Q}{4\pi\epsilon_0 r^2} \qquad (r_1 < r)$$

and the potential difference is

$$V_a - V_\infty = -\int_{r_1}^{a} \frac{Q\,dr}{4\pi\epsilon_1 r^2} - \int_{\infty}^{r_1} \frac{Q\,dr}{4\pi\epsilon_0 r^2}$$

$$= \frac{Q}{4\pi}\left[\frac{1}{\epsilon_1}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \frac{1}{\epsilon_0 r_1}\right]$$

Therefore,

$$C = \frac{4\pi}{\dfrac{1}{\epsilon_1}\left(\dfrac{1}{a} - \dfrac{1}{r_1}\right) + \dfrac{1}{\epsilon_0 r_1}} \tag{8}$$

In order to look at the problem of multiple dielectrics a little more thoroughly, let us consider a parallel-plate capacitor of area S and spacing d, with the usual assumption that d is small compared to the linear dimensions of the plates. The capacitance is $\epsilon_1 S/d$, using a dielectric of permittivity ϵ_1. Now replace a part of this dielectric by another of permittivity ϵ_2, placing the boundary between the two dielectrics parallel to the plates (Figure 6.3).

Some of us may immediately suspect that this combination is effectively two capacitors in series, yielding a total capacitance of

$$C = \frac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}$$

where $C_1 = \epsilon_1 S/d_1$ and $C_2 = \epsilon_2 S/d_2$. This is the correct result, but we can obtain it using less intuition and a more basic approach.

Because the capacitance definition, $C = Q/V$, involves a charge and a voltage, we may assume either and then find the other in terms of it. The capacitance is not a function of either, but only of the dielectrics and the geometry. Suppose we assume a potential difference V_0 between the plates. The electric field intensities in the two regions, E_2 and E_1, are both uniform, and $V_0 = E_1 d_1 + E_2 d_2$. At the dielectric interface, E is normal, and our boundary condition, Eq. (35) Chapter 5, tells us that $D_{N1} = D_{N2}$, or $\epsilon_1 E_1 = \epsilon_2 E_2$. This assumes (correctly) that there is no surface charge at the interface. Eliminating E_2 in our V_0 relation, we have

$$E_1 = \frac{V_0}{d_1 + d_2(\epsilon_1/\epsilon_2)}$$

and the surface charge density on the lower plate therefore has the magnitude

$$\rho_{S1} = D_1 = \epsilon_1 E_1 = \frac{V_0}{\dfrac{d_1}{\epsilon_1} + \dfrac{d_2}{\epsilon_2}}$$

Because $D_1 = D_2$, the magnitude of the surface charge is the same on each plate. The capacitance is then

$$C = \frac{Q}{V_0} = \frac{\rho_S S}{V_0} = \frac{1}{\dfrac{d_1}{\epsilon_1 S} + \dfrac{d_2}{\epsilon_2 S}} = \frac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}$$

As an alternate (and slightly simpler) solution, we might assume a charge Q on one plate, leading to a charge density Q/S and a value of D that is also Q/S. This is true in both regions, as $D_{N1} = D_{N2}$ and D is normal. Then $E_1 = D/\epsilon_1 = Q/(\epsilon_1 S)$, $E_2 = D/\epsilon_2 = Q/(\epsilon_2 S)$, and the potential differences across the regions are $V_1 = E_1 d_1 = Qd_1/(\epsilon_1 S)$, and $V_2 = E_2 d_2 = Qd_2/(\epsilon_2 S)$. The capacitance is

$$C = \frac{Q}{V} = \frac{Q}{V_1 + V_2} = \frac{1}{\dfrac{d_1}{\epsilon_1 S} + \dfrac{d_2}{\epsilon_2 S}} \tag{9}$$

How would the method of solution or the answer change if there were a third conducting plane along the interface? We would now expect to find surface charge on each side of this conductor, and the magnitudes of these charges should be equal. In other words, we think of the electric lines not as passing directly from one outer plate to the other, but as terminating on one side of this interior plane and then continuing on the other side. The capacitance is unchanged, provided, of course, that the added conductor is of negligible thickness. The addition of a thick conducting plate will increase the capacitance if the separation of the outer plates is kept constant, and this is an example of a more general theorem which states that the replacement of any portion of the dielectric by a conducting body will cause an increase in the capacitance.

If the dielectric boundary were placed *normal* to the two conducting plates and the dielectrics occupied areas of S_1 and S_2, then an assumed potential difference V_0 would produce field strengths $E_1 = E_2 = V_0/d$. These are tangential fields at the interface, and they must be equal. Then we may find in succession D_1, D_2, ρ_{S1}, ρ_{S2}, and Q, obtaining a capacitance

$$C = \frac{\epsilon_1 S_1 + \epsilon_2 S_2}{d} = C_1 + C_2 \tag{10}$$

as we should expect.

At this time we can do very little with a capacitor in which two dielectrics are used in such a way that the interface is not everywhere normal or parallel to the fields. Certainly we know the boundary conditions at each conductor and at the dielectric interface; however, we do not know the fields to which to apply the boundary conditions. Such a problem must be put aside until our knowledge of field theory has increased and we are willing and able to use more advanced mathematical techniques.

D6.2. Determine the capacitance of: (*a*) a 1-ft length of 35B/U coaxial cable, which has an inner conductor 0.1045 in. in diameter, a polyethylene dielectric ($\epsilon_r = 2.26$ from Table C.1), and an outer conductor that has an inner diameter of 0.680 in.; (*b*) a conducting sphere of radius 2.5 mm, covered with a polyethylene layer 2 mm thick, surrounded by a conducting sphere of radius 4.5 mm; (*c*) two rectangular conducting plates, 1 cm by 4 cm, with negligible thickness, between which are three sheets of dielectric, each 1 cm by 4 cm, and 0.1 mm thick, having dielectric constants of 1.5, 2.5, and 6.

Ans. 20.5 pF; 1.41 pF; 28.7 pF

6.4 CAPACITANCE OF A TWO-WIRE LINE

We next consider the problem of the two-wire line. The configuration consists of two parallel conducting cylinders, each of circular cross section, and we will find complete information about the electric field intensity, the potential field, the surface-charge-density distribution, and the capacitance. This arrangement is an important type of transmission line, as is the coaxial cable.

We begin by investigating the potential field of two infinite line charges. Figure 6.4 shows a positive line charge in the xz plane at $x = a$ and a negative line charge at $x = -a$. The potential of a single line charge with zero reference at a radius of R_0 is

$$V = \frac{\rho_L}{2\pi\epsilon} \ln \frac{R_0}{R}$$

We now write the expression for the combined potential field in terms of the radial distances from the positive and negative lines, R_1 and R_2, respectively,

$$V = \frac{\rho_L}{2\pi\epsilon}\left(\ln\frac{R_{10}}{R_1} - \ln\frac{R_{20}}{R_2}\right) = \frac{\rho_L}{2\pi\epsilon}\ln\frac{R_{10}R_2}{R_{20}R_1}$$

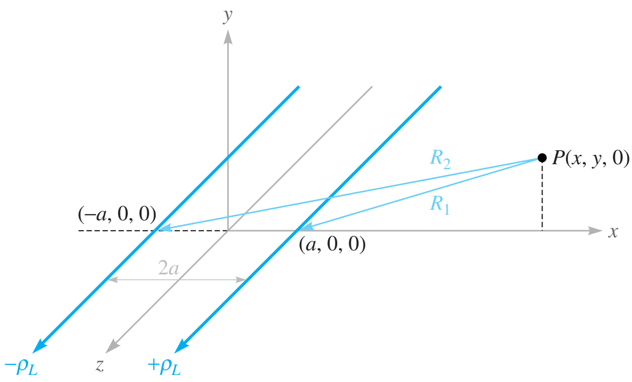

Figure 6.4 Two parallel infinite line charges carrying opposite charge. The positive line is at $x = a$, $y = 0$, and the negative line is at $x = -a$, $y = 0$. A general point $P(x, y, 0)$ in the xy plane is radially distant R_1 and R_2 from the positive and negative lines, respectively. The equipotential surfaces are circular cylinders.

We choose $R_{10} = R_{20}$, thus placing the zero reference at equal distances from each line. This surface is the $x = 0$ plane. Expressing R_1 and R_2 in terms of x and y,

$$V = \frac{\rho_L}{2\pi\epsilon}\ln\sqrt{\frac{(x+a)^2+y^2}{(x-a)^2+y^2}} = \frac{\rho_L}{4\pi\epsilon}\ln\frac{(x+a)^2+y^2}{(x-a)^2+y^2} \tag{11}$$

In order to recognize the equipotential surfaces and adequately understand the problem we are going to solve, some algebraic manipulations are necessary. Choosing an equipotential surface $V = V_1$, we define K_1 as a dimensionless parameter that is a function of the potential V_1,

$$K_1 = e^{4\pi\epsilon V_1/\rho_L} \tag{12}$$

so that

$$K_1 = \frac{(x+a)^2+y^2}{(x-a)^2+y^2}$$

After multiplying and collecting like powers, we obtain

$$x^2 - 2ax\frac{K_1+1}{K_1-1} + y^2 + a^2 = 0$$

We next work through a couple of lines of algebra and complete the square,

$$\left(x - a\frac{K_1+1}{K_1-1}\right)^2 + y^2 = \left(\frac{2a\sqrt{K_1}}{K_1-1}\right)^2$$

This shows that the $V = V_1$ equipotential surface is independent of z (or is a cylinder) and intersects the xy plane in a circle of radius b,

$$b = \frac{2a\sqrt{K_1}}{K_1-1}$$

which is centered at $x = h$, $y = 0$, where

$$h = a\frac{K_1 + 1}{K_1 - 1}$$

Now let us attack a physical problem by considering a zero-potential conducting plane located at $x = 0$, and a conducting cylinder of radius b and potential V_0 with its axis located a distance h from the plane. We solve the last two equations for a and K_1 in terms of the dimensions b and h,

$$a = \sqrt{h^2 - b^2} \tag{13}$$

and

$$\sqrt{K_1} = \frac{h + \sqrt{h^2 - b^2}}{b} \tag{14}$$

But the potential of the cylinder is V_0, so Eq. (12) leads to

$$\sqrt{K_1} = e^{2\pi\epsilon V_0/\rho_L}$$

Therefore,

$$\rho_L = \frac{4\pi\epsilon V_0}{\ln K_1} \tag{15}$$

Thus, given h, b, and V_0, we may determine a, ρ_L, and the parameter K_1. The capacitance between the cylinder and plane is now available. For a length L in the z direction, we have

$$C = \frac{\rho_L L}{V_0} = \frac{4\pi\epsilon L}{\ln K_1} = \frac{2\pi\epsilon L}{\ln\sqrt{K_1}}$$

or

$$C = \frac{2\pi\epsilon L}{\ln[(h + \sqrt{h^2 - b^2})/b]} = \frac{2\pi\epsilon L}{\cosh^{-1}(h/b)} \tag{16}$$

The solid line in Figure 6.5 shows the cross section of a cylinder of 5 m radius at a potential of 100 V in free space, with its axis 13 m distant from a plane at zero potential. Thus, $b = 5$, $h = 13$, $V_0 = 100$, and we rapidly find the location of the equivalent line charge from Eq. (13),

$$a = \sqrt{h^2 - b^2} = \sqrt{13^2 - 5^2} = 12\,\text{m}$$

the value of the potential parameter K_1 from Eq. (14),

$$\sqrt{K_1} = \frac{h + \sqrt{h^2 - b^2}}{b} = \frac{13 + 12}{5} = 5 \qquad K_1 = 25$$

the strength of the equivalent line charge from Eq. (15),

$$\rho_L = \frac{4\pi\epsilon V_0}{\ln K_1} = \frac{4\pi \times 8.854 \times 10^{-12} \times 100}{\ln 25} = 3.46\ \text{nC/m}$$

and the capacitance between cylinder and plane from Eq. (16),

$$C = \frac{2\pi\epsilon}{\cosh^{-1}(h/b)} = \frac{2\pi \times 8.854 \times 10^{-12}}{\cosh^{-1}(13/5)} = 34.6\ \text{pF/m}$$

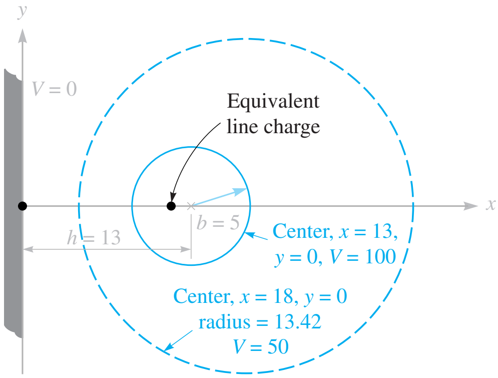

Figure 6.5 A numerical example of the capacitance, linear charge density, position of an equivalent line charge, and characteristics of the mid-equipotential surface for a cylindrical conductor of 5 m radius at a potential of 100 V, parallel to and 13 m from a conducting plane at zero potential.

We may also identify the cylinder representing the 50 V equipotential surface by finding new values for K_1, h, and b. We first use Eq. (12) to obtain

$$K_1 = e^{4\pi\epsilon V_1/\rho_L} = e^{4\pi\times 8.854\times 10^{-12}\times 50/3.46\times 10^{-9}} = 5.00$$

Then the new radius is

$$b = \frac{2a\sqrt{K_1}}{K_1 - 1} = \frac{2\times 12\sqrt{5}}{5-1} = 13.42 \text{ m}$$

and the corresponding value of h becomes

$$h = a\frac{K_1 + 1}{K_1 - 1} = 12\frac{5+1}{5-1} = 18 \text{ m}$$

This cylinder is shown in color in Figure 6.5.

The electric field intensity can be found by taking the gradient of the potential field, as given by Eq. (11),

$$\mathbf{E} = -\nabla\left[\frac{\rho_L}{4\pi\epsilon}\ln\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}\right]$$

Thus,

$$\mathbf{E} = -\frac{\rho_L}{4\pi\epsilon}\left[\frac{2(x+a)\mathbf{a}_x + 2y\mathbf{a}_y}{(x+a)^2 + y^2} - \frac{2(x-a)\mathbf{a}_x + 2y\mathbf{a}_y}{(x-a)^2 + y^2}\right]$$

and

$$\mathbf{D} = \epsilon\mathbf{E} = -\frac{\rho_L}{2\pi}\left[\frac{(x+a)\mathbf{a}_x + y\mathbf{a}_y}{(x+a)^2 + y^2} - \frac{(x-a)\mathbf{a}_x + y\mathbf{a}_y}{(x-a)^2 + y^2}\right]$$

If we evaluate D_x at $x = h - b$, $y = 0$, we may obtain $\rho_{S,\max}$

$$\rho_{S,\max} = -D_{x,x=h-b,y=0} = \frac{\rho_L}{2\pi}\left[\frac{h-b+a}{(h-b+a)^2} - \frac{h-b-a}{(h-b-a)^2}\right]$$

For our example,

$$\rho_{S,\max} = \frac{3.46 \times 10^{-9}}{2\pi}\left[\frac{13-5+12}{(13-5+12)^2} - \frac{13-5-12}{(13-5-12)^2}\right] = 0.165 \text{ nC/m}^2$$

Similarly, $\rho_{S,\min} = D_{x,x=h+b,y=0}$, and

$$\rho_{S,\min} = \frac{3.46 \times 10^{-9}}{2\pi}\left[\frac{13+5+12}{30^2} - \frac{13+5-12}{6^2}\right] = 0.073 \text{ nC/m}^2$$

Thus,

$$\rho_{S,\max} = 2.25\rho_{S,\min}$$

If we apply Eq. (16) to the case of a conductor for which $b \ll h$, then

$$\ln\left[\left(h + \sqrt{h^2 - b^2}\right)/b\right] \doteq \ln[(h+h)/b] \doteq \ln(2h/b)$$

and

$$C = \frac{2\pi\epsilon L}{\ln(2h/b)} \qquad (b \ll h) \tag{17}$$

The capacitance between two circular conductors separated by a distance $2h$ is one-half the capacitance given by Eqs. (16) or (17). This last answer is of interest because it gives us an expression for the capacitance of a section of two-wire transmission line, one of the types of transmission lines studied later in Chapter 13.

D6.3. A conducting cylinder with a radius of 1 cm and at a potential of 20 V is parallel to a conducting plane which is at zero potential. The plane is 5 cm distant from the cylinder axis. If the conductors are embedded in a perfect dielectric for which $\epsilon_r = 4.5$, find: (*a*) the capacitance per unit length between cylinder and plane; (*b*) $\rho_{S,\max}$ on the cylinder.

Ans. 109.2 pF/m; 42.6 nC/m^2

6.5 USING FIELD SKETCHES TO ESTIMATE CAPACITANCE IN TWO-DIMENSIONAL PROBLEMS

In capacitance problems in which the conductor configurations cannot be easily described using a single coordinate system, other analysis techniques are usually applied. Such methods typically involve a numerical determination of field or potential values over a grid within the region of interest. In this section, another approach is described that involves making sketches of field lines and equipotential surfaces in a manner that follows a few simple rules. This approach, although lacking the accuracy of more elegant methods, allows fairly quick estimates of capacitance while providing a useful visualization of the field configuration.

The method, requiring only pencil and paper, is capable of yielding good accuracy if used skillfully and patiently. Fair accuracy (5 to 10 percent on a capacitance determination) may be obtained by a beginner who does no more than follow the few rules and hints of the art. The method to be described is applicable only to fields in which no variation exists in the direction normal to the plane of the sketch. The procedure is based on several facts that we have already demonstrated:

1. A conductor boundary is an equipotential surface.
2. The electric field intensity and electric flux density are both perpendicular to the equipotential surfaces.
3. **E** and **D** are therefore perpendicular to the conductor boundaries and possess zero tangential values.
4. The lines of electric flux, or streamlines, begin and terminate on charge and hence, in a charge-free, homogeneous dielectric, begin and terminate only on the conductor boundaries.

We consider the implications of these statements by drawing the streamlines on a sketch that already shows the equipotential surfaces. In Figure 6.6*a*, two conductor boundaries are shown, and equipotentials are drawn with a constant potential difference between lines. We should remember that these lines are only the cross sections of the equipotential surfaces, which are cylinders (although not circular). No variation in the direction normal to the surface of the paper is permitted. We arbitrarily choose to begin a streamline, or flux line, at A on the surface of the more positive conductor. It leaves the surface normally and must cross at right angles the undrawn but very real equipotential surfaces between the conductor and the first surface shown. The line is continued to the other conductor, obeying the single rule that the intersection with each equipotential must be square.

In a similar manner, we may start at B and sketch another streamline ending at B'. We need to understand the meaning of this pair of streamlines. The streamline,

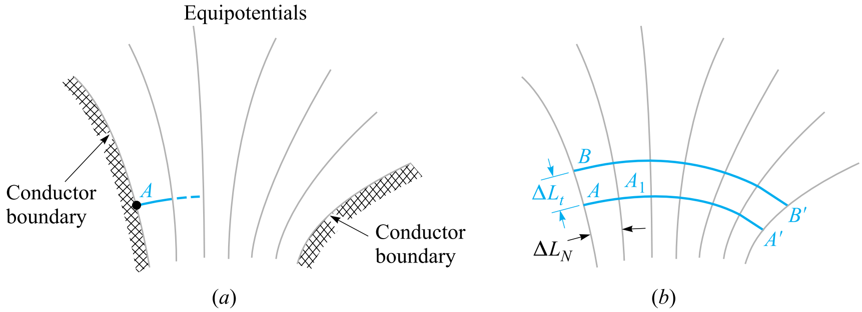

Figure 6.6 (*a*) Sketch of the equipotential surfaces between two conductors. The increment of potential between each of the two adjacent equipotentials is the same. (*b*) One flux line has been drawn from A to A', and a second from B to B'.

by definition, is everywhere tangent to the electric field intensity or to the electric flux density. Because the streamline is tangent to the electric flux density, the flux density is tangent to the streamline, and no electric flux may cross any streamline. In other words, if there is a charge of 5 μC on the surface between A and B (and extending 1 m into the paper), then 5 μC of flux begins in this region, and all must terminate between A' and B'. Such a pair of lines is sometimes called a flux *tube*, because it physically seems to carry flux from one conductor to another without losing any.

We next construct a third streamline, and both the mathematical and visual interpretations we may make from the sketch will be greatly simplified if we draw this line starting from some point C chosen so that the same amount of flux is carried in the tube BC as is contained in AB. How do we choose the position of C?

The electric field intensity at the midpoint of the line joining A to B may be found approximately by assuming a value for the flux in the tube AB, say $\Delta\Psi$, which allows us to express the electric flux density by $\Delta\Psi/\Delta L_t$, where the depth of the tube into the paper is 1 m and ΔL_t is the length of the line joining A to B. The magnitude of E is then

$$E = \frac{1}{\epsilon}\frac{\Delta\Psi}{\Delta L_t}$$

We may also find the magnitude of the electric field intensity by dividing the potential difference between points A and A_1, lying on two adjacent equipotential surfaces, by the distance from A to A_1. If this distance is designated ΔL_N and an increment of potential between equipotentials of ΔV is assumed, then

$$E = \frac{\Delta V}{\Delta L_N}$$

This value applies most accurately to the point at the middle of the line segment from A to A_1, while the previous value was most accurate at the midpoint of the line segment from A to B. If, however, the equipotentials are close together (ΔV small) and the two streamlines are close together ($\Delta\Psi$ small), the two values found for the electric field intensity must be approximately equal,

$$\frac{1}{\epsilon}\frac{\Delta\Psi}{\Delta L_t} = \frac{\Delta V}{\Delta L_N} \tag{18}$$

Throughout our sketch we have assumed a homogeneous medium (ϵ constant), a constant increment of potential between equipotentials (ΔV constant), and a constant amount of flux per tube ($\Delta\Psi$ constant). To satisfy all these conditions, Eq. (18) shows that

$$\frac{\Delta L_t}{\Delta L_N} = \text{constant} = \frac{1}{\epsilon}\frac{\Delta\Psi}{\Delta V} \tag{19}$$

A similar argument might be made at any point in our sketch, and we are therefore led to the conclusion that a constant ratio must be maintained between the distance between streamlines as measured along an equipotential, and the distance between equipotentials as measured along a streamline. It is this *ratio* that must have the same value at every point, not the individual lengths. Each length must decrease in regions of greater field strength, because ΔV is constant.

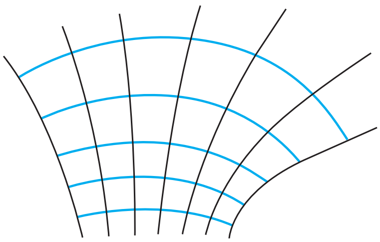

Figure 6.7 The remaining of the streamlines have been added to Fig. 6.6*b* by beginning each new line normally to the conductor and maintaining curvilinear squares throughout the sketch.

The simplest ratio we can use is unity, and the streamline from B to B' shown in Figure 6.6*b* was started at a point for which $\Delta L_t = \Delta L_N$. Because the ratio of these distances is kept at unity, the streamlines and equipotentials divide the field-containing region into curvilinear squares, a term implying a planar geometric figure that differs from a true square in having slightly curved and slightly unequal sides but which approaches a square as its dimensions decrease. Those incremental surface elements in our three coordinate systems which are planar may also be drawn as curvilinear squares.

We may now sketch in the remainder of the streamlines by keeping each small box as square as possible. One streamline is begun, an equipotential line is roughed in, another streamline is added, forming a curvilinear square, and the map is gradually extended throughout the desired region. The complete sketch is shown in Figure 6.7.

The construction of a useful field map is an art; the science merely furnishes the rules. Proficiency in any art requires practice. A good problem for beginners is the coaxial cable or coaxial capacitor, since all the equipotentials are circles while the flux lines are straight lines. The next sketch attempted should be two parallel circular conductors, where the equipotentials are again circles but with different centers. Each of these is given as a problem at the end of the chapter.

Figure 6.8 shows a completed map for a cable containing a square inner conductor surrounded by a circular conductor. The capacitance is found from $C = Q/V_0$ by replacing Q by $N_Q \Delta Q = N_Q \Delta\Psi$, where N_Q is the number of flux tubes joining the two conductors, and letting $V_0 = N_V \Delta V$, where N_V is the number of potential increments between conductors,

$$C = \frac{N_Q \Delta Q}{N_V \Delta V}$$

and then using Eq. (19),

$$C = \frac{N_Q}{N_V} \epsilon \frac{\Delta L_t}{\Delta L_N} = \epsilon \frac{N_Q}{N_V} \tag{20}$$

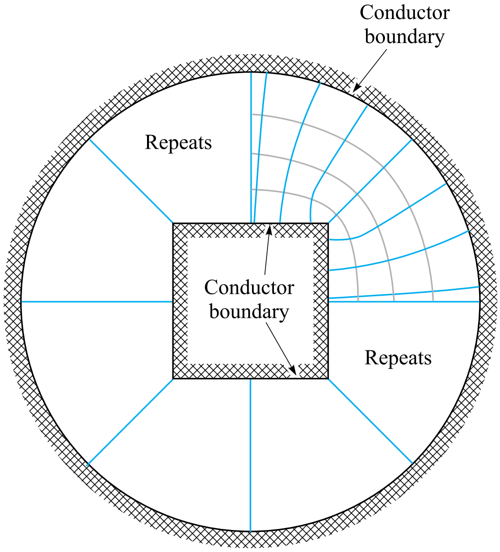

Figure 6.8 An example of a curvilinear-square field map. The side of the square is two-thirds the radius of the circle. $N_V = 4$ and $N_Q = 8 \times 3.25 \times 26$, and therefore $C = \epsilon_0 N_Q / N_V = 57.6$ pF/m.

since $\Delta L_t / \Delta L_N = 1$. The determination of the capacitance from a flux plot merely consists of counting squares in two directions, between conductors and around either conductor. From Figure 6.8 we obtain

$$C = \epsilon_0 \frac{8 \times 3.25}{4} = 57.6 \text{ pF/m}$$

Ramo, Whinnery, and Van Duzer have an excellent discussion with examples of the construction of field maps by curvilinear squares. They offer the following suggestions:[1]

1. Plan on making a number of rough sketches, taking only a minute or so apiece, before starting any plot to be made with care. The use of transparent paper over the basic boundary will speed up this preliminary sketching.
2. Divide the known potential difference between electrodes into an equal number of divisions, say four or eight to begin with.
3. Begin the sketch of equipotentials in the region where the field is known best, for example, in some region where it approaches a uniform field. Extend the equipotentials according to your best guess throughout the plot. Note that they should tend to hug acute angles of the conducting boundary and be spread out in the vicinity of obtuse angles of the boundary.

[1] By permission from S. Ramo, J. R. Whinnery, and T. Van Duzer, pp. 51–52. See References at the end of this chapter. Curvilinear maps are discussed on pp. 50–52.

4. Draw in the orthogonal set of field lines. As these are started, they should form curvilinear squares, but, as they are extended, the condition of orthogonality should be kept paramount, even though this will result in some rectangles with ratios other than unity.
5. Look at the regions with poor side ratios and try to see what was wrong with the first guess of equipotentials. Correct them and repeat the procedure until reasonable curvilinear squares exist throughout the plot.
6. In regions of low field intensity, there will be large figures, often of five or six sides. To judge the correctness of the plot in this region, these large units should be subdivided. The subdivisions should be started back away from the region needing subdivision, and each time a flux tube is divided in half, the potential divisions in this region must be divided by the same factor.

D6.4. Figure 6.9 shows the cross section of two circular cylinders at potentials of 0 and 60 V. The axes are parallel and the region between the cylinders is air-filled. Equipotentials at 20 V and 40 V are also shown. Prepare a curvilinear-square map on the figure and use it to establish suitable values for: (*a*) the capacitance per meter length; (*b*) E at the left side of the 60 V conductor if its true radius is 2 mm; (*c*) ρ_S at that point.

Ans. 69 pF/m; 60 kV/m; 550 nC/m^2

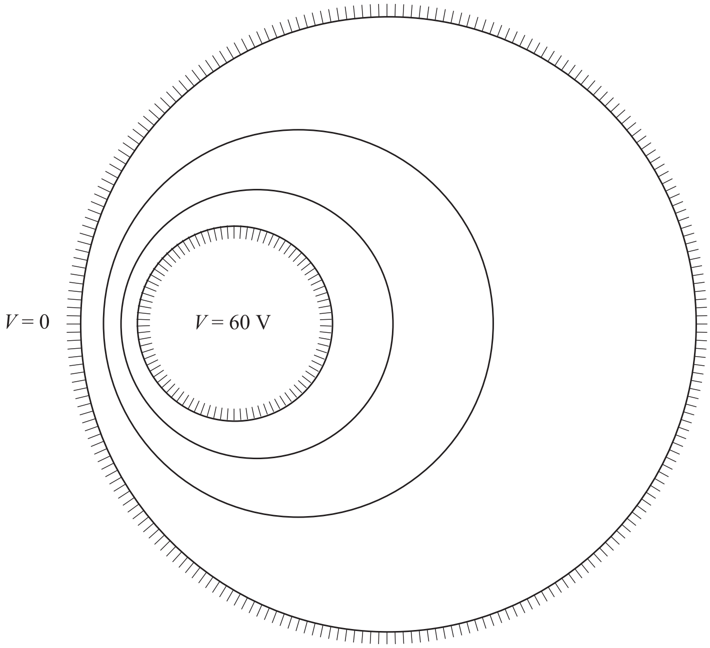

Figure 6.9 See Problem D6.4.

6.6 POISSON'S AND LAPLACE'S EQUATIONS

In preceding sections, we have found capacitance by first assuming a known charge distribution on the conductors and then finding the potential difference in terms of the assumed charge. An alternate approach would be to start with known potentials on each conductor, and then work backward to find the charge in terms of the known potential difference. The capacitance in either case is found by the ratio Q/V.

The first objective in the latter approach is thus to find the potential function between conductors, given values of potential on the boundaries, along with possible volume charge densities in the region of interest. The mathematical tools that enable this to happen are Poisson's and Laplace's equations, to be explored in the remainder of this chapter. Problems involving one to three dimensions can be solved either analytically or numerically. Laplace's and Poisson's equations, when compared to other methods, are probably the most widely useful because many problems in engineering practice involve devices in which applied potential differences are known, and in which constant potentials occur at the boundaries.

Obtaining Poisson's equation is exceedingly simple, for from the point form of Gauss's law,

$$\nabla \cdot \mathbf{D} = \rho_\nu \tag{21}$$

the definition of **D**,

$$\mathbf{D} = \epsilon \mathbf{E} \tag{22}$$

and the gradient relationship,

$$\mathbf{E} = -\nabla V \tag{23}$$

by substitution we have

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = -\nabla \cdot (\epsilon \nabla V) = \rho_\nu$$

or

$$\nabla \cdot \nabla V = -\frac{\rho_\nu}{\epsilon} \tag{24}$$

for a homogeneous region in which ϵ is constant.

Equation (24) is *Poisson's equation,* but the "double ∇" operation must be interpreted and expanded, at least in rectangular coordinates, before the equation can be useful. In rectangular coordinates,

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

and therefore

$$\nabla \cdot \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{25}$$

Usually the operation $\nabla \cdot \nabla$ is abbreviated ∇^2 (and pronounced "del squared"), a good reminder of the second-order partial derivatives appearing in Eq. (5), and we have

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_\nu}{\epsilon}} \tag{26}$$

in rectangular coordinates.

If $\rho_\nu = 0$, indicating zero *volume* charge density, but allowing point charges, line charge, and surface charge density to exist at singular locations as sources of the field, then

$$\boxed{\nabla^2 V = 0} \tag{27}$$

which is *Laplace's equation*. The ∇^2 operation is called the *Laplacian of V*.

In rectangular coordinates Laplace's equation is

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{(rectangular)}} \tag{28}$$

and the form of $\nabla^2 V$ in cylindrical and spherical coordinates may be obtained by using the expressions for the divergence and gradient already obtained in those coordinate systems. For reference, the Laplacian in cylindrical coordinates is

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \qquad \text{(cylindrical)}} \tag{29}$$

and in spherical coordinates is

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \qquad \text{(spherical)}} \tag{30}$$

These equations may be expanded by taking the indicated partial derivatives, but it is usually more helpful to have them in the forms just given; furthermore, it is much easier to expand them later if necessary than it is to put the broken pieces back together again.

Laplace's equation is all-embracing, for, applying as it does wherever volume charge density is zero, it states that every conceivable configuration of electrodes or conductors produces a field for which $\nabla^2 V = 0$. All these fields are different, with different potential values and different spatial rates of change, yet for each of them $\nabla^2 V = 0$. Because *every* field (if $\rho_\nu = 0$) satisfies Laplace's equation, how can we expect to reverse the procedure and use Laplace's equation to find one specific field in which we happen to have an interest? Obviously, more information is required, and we shall find that we must solve Laplace's equation subject to certain *boundary conditions*.

Every physical problem must contain at least one conducting boundary and usually contains two or more. The potentials on these boundaries are assigned values, perhaps $V_0, V_1, \ldots$, or perhaps numerical values. These definite equipotential surfaces will provide the boundary conditions for the type of problem to be solved. In other types of problems, the boundary conditions take the form of specified values of E (alternatively, a surface charge density, ρ_S) on an enclosing surface, or a mixture of known values of V and E.

Before using Laplace's equation or Poisson's equation in several examples, we must state that if our answer satisfies Laplace's equation and also satisfies the boundary conditions, then it is the only possible answer. This is a statement of the Uniqueness Theorem, the proof of which is presented in Appendix D.

D6.5. Calculate numerical values for V and ρ_ν at point P in free space if: (a) $V = \dfrac{4yz}{x^2+1}$, at $P(1, 2, 3)$; (b) $V = 5\rho^2 \cos 2\phi$, at $P(\rho = 3, \phi = \dfrac{\pi}{3}, z = 2)$; (c) $V = \dfrac{2\cos\phi}{r^2}$, at $P(r = 0.5, \theta = 45^\circ, \phi = 60^\circ)$.

Ans. 12 V, −106.2 pC/m^3; −22.5 V, 0; 4 V, 0

6.7 EXAMPLES OF THE SOLUTION OF LAPLACE'S EQUATION

Several methods have been developed for solving Laplace's equation. The simplest method is that of direct integration. We will use this technique to work several examples involving one-dimensional potential variation in various coordinate systems in this section.

The method of direct integration is applicable only to problems that are "one-dimensional," or in which the potential field is a function of only one of the three coordinates. Since we are working with only three coordinate systems, it might seem, then, that there are nine problems to be solved, but a little reflection will show that a field that varies only with x is fundamentally the same as a field that varies only with y. Rotating the physical problem a quarter turn is no change. Actually, there are only five problems to be solved, one in rectangular coordinates, two in cylindrical, and two in spherical. We will solve them all.

First, let us assume that V is a function only of x and worry later about which physical problem we are solving when we have a need for boundary conditions. Laplace's equation reduces to

$$\frac{\partial^2 V}{\partial x^2} = 0$$

and the partial derivative may be replaced by an ordinary derivative, since V is not a function of y or z,

$$\frac{d^2V}{dx^2} = 0$$

We integrate twice, obtaining

$$\frac{dV}{dx} = A$$

and

$$V = Ax + B \tag{31}$$

where A and B are constants of integration. Equation (31) contains two such constants, as we would expect for a second-order differential equation. These constants can be determined only from the boundary conditions.

Since the field varies only with x and is not a function of y and z, then V is a constant if x is a constant or, in other words, the equipotential surfaces are parallel planes normal to the x axis. The field is thus that of a parallel-plate capacitor, and as soon as we specify the potential on any two planes, we may evaluate our constants of integration.

EXAMPLE 6.2

Start with the potential function, Eq. (31), and find the capacitance of a parallel-plate capacitor of plate area S, plate separation d, and potential difference V_0 between plates.

Solution. Take $V = 0$ at $x = 0$ and $V = V_0$ at $x = d$. Then from Eq. (31),

$$A = \frac{V_0}{d} \qquad B = 0$$

and

$$\boxed{V = \frac{V_0 x}{d}} \tag{32}$$

We still need the total charge on either plate before the capacitance can be found. We should remember that when we first solved this capacitor problem, the sheet of charge provided our starting point. We did not have to work very hard to find the charge, for all the fields were expressed in terms of it. The work then was spent in finding potential difference. Now the problem is reversed (and simplified).

The necessary steps are these, after the choice of boundary conditions has been made:

1. Given V, use $\mathbf{E} = -\nabla V$ to find $\mathbf{E}$.
2. Use $\mathbf{D} = \epsilon\mathbf{E}$ to find $\mathbf{D}$.
3. Evaluate $\mathbf{D}$ at either capacitor plate, $\mathbf{D} = \mathbf{D}_S = D_N \mathbf{a}_N$.
4. Recognize that $\rho_S = D_N$.
5. Find Q by a surface integration over the capacitor plate, $Q = \int_S \rho_S \, dS$.

Here we have

$$V = V_0 \frac{x}{d}$$

$$\mathbf{E} = -\frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{D} = -\epsilon \frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{D}_S = \mathbf{D}\Big|_{x=0} = -\epsilon \frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{a}_N = \mathbf{a}_x$$

$$D_N = -\epsilon \frac{V_0}{d} = \rho_S$$

$$Q = \int_S \frac{-\epsilon V_0}{d} dS = -\epsilon \frac{V_0 S}{d}$$

and the capacitance is

$$C = \frac{|Q|}{V_0} = \frac{\epsilon S}{d} \tag{33}$$

We will use this procedure several times in the examples to follow.

EXAMPLE 6.3

Because no new problems are solved by choosing fields which vary only with y or with z in rectangular coordinates, we pass on to cylindrical coordinates for our next example. Variations with respect to z are again nothing new, and we next assume variation with respect to ρ only. Laplace's equation becomes

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

Noting the ρ in the denominator, we exclude $\rho = 0$ from our solution and then multiply by ρ and integrate,

$$\rho \frac{dV}{d\rho} = A$$

where a total derivative replaces the partial derivative because V varies only with ρ. Next, rearrange, and integrate again,

$$V = A \ln \rho + B \tag{34}$$

The equipotential surfaces are given by $\rho =$ constant and are cylinders, and the problem is that of the coaxial capacitor or coaxial transmission line. We choose a potential difference of V_0 by letting $V = V_0$ at $\rho = a$, $V = 0$ at $\rho = b$, $b > a$, and obtain

$$V = V_0 \frac{\ln(b/\rho)}{\ln(b/a)} \tag{35}$$

from which

$$\mathbf{E} = \frac{V_0}{\rho} \frac{1}{\ln(b/a)} \mathbf{a}_\rho$$

$$D_{N(\rho=a)} = \frac{\epsilon V_0}{a \ln(b/a)}$$

$$Q = \frac{\epsilon V_0 2\pi a L}{a \ln(b/a)}$$

$$C = \frac{2\pi \epsilon L}{\ln(b/a)} \tag{36}$$

which agrees with our result in Section 6.3 (Eq. (5)).

EXAMPLE 6.4

Now assume that V is a function only of ϕ in cylindrical coordinates. We might look at the physical problem first for a change and see that equipotential surfaces are given by $\phi =$ constant. These are radial planes. Boundary conditions might be $V = 0$ at $\phi = 0$ and $V = V_0$ at $\phi = \alpha$, leading to the physical problem detailed in Figure 6.10.

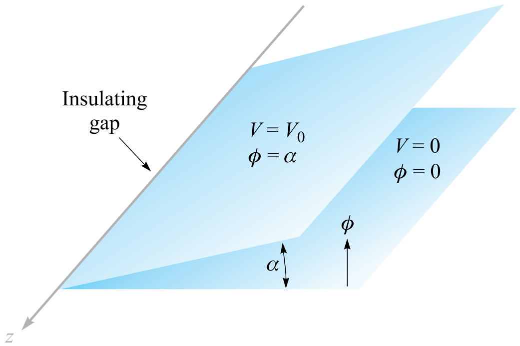

Figure 6.10 Two infinite radial planes with an interior angle α. An infinitesimal insulating gap exists at $\rho = 0$. The potential field may be found by applying Laplace's equation in cylindrical coordinates.

Laplace's equation is now

$$\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

We exclude $\rho = 0$ and have

$$\frac{d^2 V}{d\phi^2} = 0$$

The solution is

$$V = A\phi + B$$

The boundary conditions determine A and B, and

$$V = V_0 \frac{\phi}{\alpha} \tag{37}$$

Taking the gradient of Eq. (37) produces the electric field intensity,

$$\mathbf{E} = -\frac{V_0 \mathbf{a}_\phi}{\alpha\rho} \tag{38}$$

and it is interesting to note that E is a function of ρ and not of ϕ. This does not contradict our original assumptions, which were restrictions only on the potential field. Note, however, that the *vector* field $\mathbf{E}$ is in the ϕ direction.

A problem involving the capacitance of these two radial planes is included at the end of the chapter.

EXAMPLE 6.5

We now turn to spherical coordinates, dispose immediately of variations with respect to ϕ only as having just been solved, and treat first $V = V(r)$.

The details are left for a problem later, but the final potential field is given by

$$V = V_0 \frac{\frac{1}{r} - \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}} \tag{39}$$

where the boundary conditions are evidently $V = 0$ at $r = b$ and $V = V_0$ at $r = a$, $b > a$. The problem is that of concentric spheres. The capacitance was found previously in Section 6.3 (by a somewhat different method) and is

$$C = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}} \tag{40}$$

EXAMPLE 6.6

In spherical coordinates we now restrict the potential function to $V = V(\theta)$, obtaining

$$\frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) = 0$$

We exclude $r = 0$ and $\theta = 0$ or π and have

$$\sin\theta\frac{dV}{d\theta} = A$$

The second integral is then

$$V = \int \frac{A\,d\theta}{\sin\theta} + B$$

which is not as obvious as the previous ones. From integral tables (or a good memory) we have

$$V = A\ln\left(\tan\frac{\theta}{2}\right) + B \qquad (41)$$

The equipotential surfaces of Eq. (41) are cones. Figure 6.11 illustrates the case where $V = 0$ at $\theta = \pi/2$ and $V = V_0$ at $\theta = \alpha$, $\alpha < \pi/2$. We obtain

$$V = V_0 \frac{\ln\left(\tan\frac{\theta}{2}\right)}{\ln\left(\tan\frac{\alpha}{2}\right)} \qquad (42)$$

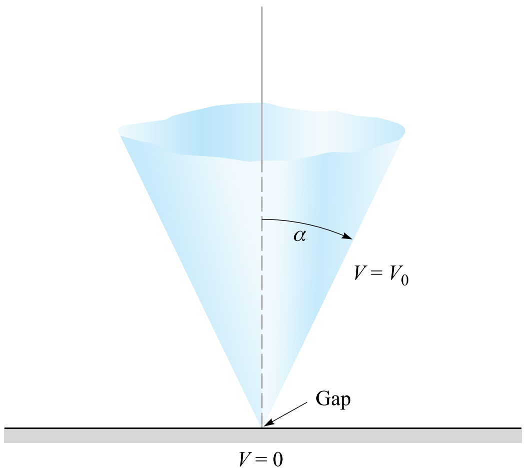

Figure 6.11 For the cone $\theta = \alpha$ at V_0 and the plane $\theta = \pi/2$ at $V = 0$, the potential field is given by $V = V_0[\ln(\tan\theta/2)]/[\ln(\tan\alpha/2)]$.

In order to find the capacitance between a conducting cone with its vertex separated from a conducting plane by an infinitesimal insulating gap and its axis normal to the plane, we first find the field strength:

$$\mathbf{E} = -\nabla V = \frac{-1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta = -\frac{V_0}{r \sin\theta \ln\left(\tan\dfrac{\alpha}{2}\right)}\mathbf{a}_\theta$$

The surface charge density on the cone is then

$$\rho_S = \frac{-\epsilon V_0}{r \sin\alpha \ln\left(\tan\dfrac{\alpha}{2}\right)}$$

producing a total charge Q,

$$Q = \frac{-\epsilon V_0}{\sin\alpha \ln\left(\tan\dfrac{\alpha}{2}\right)} \int_0^\infty \int_0^{2\pi} \frac{r \sin\alpha \, d\phi \, dr}{r}$$

$$= \frac{-2\pi\epsilon_0 V_0}{\ln\left(\tan\dfrac{\alpha}{2}\right)} \int_0^\infty dr$$

This leads to an infinite value of charge and capacitance, and it becomes necessary to consider a cone of finite size. Our answer will now be only an approximation because the theoretical equipotential surface is $\theta = \alpha$, a conical surface extending from $r = 0$ to $r = \infty$, whereas our physical conical surface extends only from $r = 0$ to, say, $r = r_1$. The approximate capacitance is

$$C \doteq \frac{2\pi\epsilon r_1}{\ln\left(\cot\dfrac{\alpha}{2}\right)} \tag{43}$$

If we desire a more accurate answer, we may make an estimate of the capacitance of the base of the cone to the zero-potential plane and add this amount to our answer. Fringing, or nonuniform, fields in this region have been neglected and introduce an additional source of error.

D6.6. Find $|\mathbf{E}|$ at $P(3, 1, 2)$ in rectangular coordinates for the field of: (*a*) two coaxial conducting cylinders, $V = 50$ V at $\rho = 2$ m, and $V = 20$ V at $\rho = 3$ m; (*b*) two radial conducting planes, $V = 50$ V at $\phi = 10°$, and $V = 20$ V at $\phi = 30°$.

Ans. 23.4 V/m; 27.2 V/m

6.8 EXAMPLE OF THE SOLUTION OF POISSON'S EQUATION: THE P-N JUNCTION CAPACITANCE

To select a reasonably simple problem that might illustrate the application of Poisson's equation, we must assume that the volume charge density is specified. This is not usually the case, however; in fact, it is often the quantity about which we are seeking further information. The type of problem which we might encounter later would begin with a knowledge only of the boundary values of the potential, the electric field intensity, and the current density. From these we would have to apply Poisson's equation, the continuity equation, and some relationship expressing the forces on the charged particles, such as the Lorentz force equation or the diffusion equation, and solve the whole system of equations simultaneously. Such an ordeal is beyond the scope of this text, and we will therefore assume a reasonably large amount of information.

As an example, let us select a *pn* junction between two halves of a semiconductor bar extending in the x direction. We will assume that the region for $x < 0$ is doped p type and that the region for $x > 0$ is n type. The degree of doping is identical on each side of the junction. To review some of the facts about the semiconductor junction, we note that initially there are excess holes to the left of the junction and excess electrons to the right. Each diffuses across the junction until an electric field is built up in such a direction that the diffusion current drops to zero. Thus, to prevent more holes from moving to the right, the electric field in the neighborhood of the junction must be directed to the left; E_x is negative there. This field must be produced by a net positive charge to the right of the junction and a net negative charge to the left. Note that the layer of positive charge consists of two parts—the holes which have crossed the junction and the positive donor ions from which the electrons have departed. The negative layer of charge is constituted in the opposite manner by electrons and negative acceptor ions.

The type of charge distribution that results is shown in Figure 6.12*a*, and the negative field which it produces is shown in Figure 6.12*b*. After looking at these two figures, one might profitably read the previous paragraph again.

A charge distribution of this form may be approximated by many different expressions. One of the simpler expressions is

$$\rho_v = 2\rho_{v0} \operatorname{sech} \frac{x}{a} \tanh \frac{x}{a} \tag{44}$$

which has a maximum charge density $\rho_{v,max} = \rho_{v0}$ that occurs at $x = 0.881a$. The maximum charge density ρ_{v0} is related to the acceptor and donor concentrations N_a and N_d by noting that all the donor and acceptor ions in this region (the *depletion* layer) have been stripped of an electron or a hole, and thus

$$\rho_{v0} = eN_a = eN_d$$

We now solve Poisson's equation,

$$\nabla^2 V = -\frac{\rho_v}{\epsilon}$$

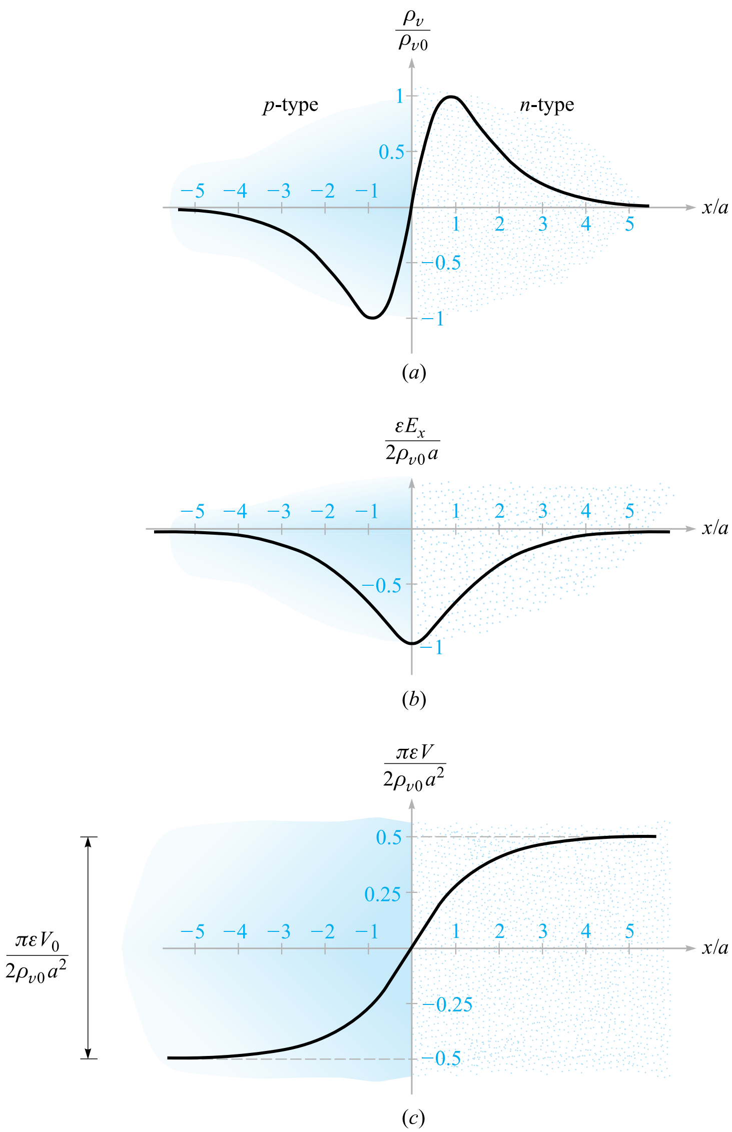

Figure 6.12 (*a*) The charge density, (*b*) the electric field intensity, and (*c*) the potential are plotted for a *pn* junction as functions of distance from the center of the junction. The *p*-type material is on the left, and the *n*-type is on the right.

subject to the charge distribution assumed above,

$$\frac{d^2V}{dx^2} = -\frac{2\rho_{v0}}{\epsilon}\text{sech}\frac{x}{a}\tanh\frac{x}{a}$$

in this one-dimensional problem in which variations with y and z are not present. We integrate once,

$$\frac{dV}{dx} = \frac{2\rho_{v0}a}{\epsilon}\text{sech}\frac{x}{a} + C_1$$

and obtain the electric field intensity,

$$E_x = -\frac{2\rho_{v0}a}{\epsilon}\text{sech}\frac{x}{a} - C_1$$

To evaluate the constant of integration C_1, we note that no net charge density and no fields can exist *far* from the junction. Thus, as $x \to \pm\infty$, E_x must approach zero. Therefore $C_1 = 0$, and

$$E_x = -\frac{2\rho_{v0}a}{\epsilon}\text{sech}\frac{x}{a} \tag{45}$$

Integrating again,

$$V = \frac{4\rho_{v0}a^2}{\epsilon}\tan^{-1}e^{x/a} + C_2$$

Let us arbitrarily select our zero reference of potential at the center of the junction, $x = 0$,

$$0 = \frac{4\rho_{v0}a^2}{\epsilon}\frac{\pi}{4} + C_2$$

and finally,

$$V = \frac{4\rho_{v0}a^2}{\epsilon}\left(\tan^{-1}e^{x/a} - \frac{\pi}{4}\right) \tag{46}$$

Figure 6.12 shows the charge distribution (a), electric field intensity (b), and the potential (c), as given by Eqs. (44), (45), and (46), respectively.

The potential is constant once we are a distance of about $4a$ or $5a$ from the junction. The total potential difference V_0 across the junction is obtained from Eq. (46),

$$V_0 = V_{x\to\infty} - V_{x\to-\infty} = \frac{2\pi\rho_{v0}a^2}{\epsilon} \tag{47}$$

This expression suggests the possibility of determining the total charge on one side of the junction and then using Eq. (47) to find a junction capacitance. The total positive charge is

$$Q = S\int_0^\infty 2\rho_{v0}\text{sech}\frac{x}{a}\tanh\frac{x}{a}\,dx = 2\rho_{v0}aS$$

where S is the area of the junction cross section. If we make use of Eq. (47) to eliminate the distance parameter a, the charge becomes

$$Q = S\sqrt{\frac{2\rho_{v0}\epsilon V_0}{\pi}} \tag{48}$$

Because the total charge is a function of the potential difference, we have to be careful in defining a capacitance. Thinking in "circuit" terms for a moment,

$$I = \frac{dQ}{dt} = C\frac{dV_0}{dt}$$

and thus

$$C = \frac{dQ}{dV_0}$$

By differentiating Eq. (48), we therefore have the capacitance

$$C = \sqrt{\frac{\rho_{\nu 0}\epsilon}{2\pi V_0}}\,S = \frac{\epsilon S}{2\pi a} \tag{49}$$

The first form of Eq. (49) shows that the capacitance varies inversely as the square root of the voltage. That is, a higher voltage causes a greater separation of the charge layers and a smaller capacitance. The second form is interesting in that it indicates that we may think of the junction as a parallel-plate capacitor with a "plate" separation of $2\pi a$. In view of the dimensions of the region in which the charge is concentrated, this is a logical result.

Poisson's equation enters into any problem involving volume charge density. Besides semiconductor diode and transistor models, we find that vacuum tubes, magnetohydrodynamic energy conversion, and ion propulsion require its use in constructing satisfactory theories.

D6.7. In the neighborhood of a certain semiconductor junction, the volume charge density is given by $\rho_\nu = 750 \text{ sech } 10^6\pi x \tanh\ 10^6\pi x$ C/m^3. The dielectric constant of the semiconductor material is 10 and the junction area is 2×10^{-7} m^2. Find: (*a*) V_0; (*b*) C; (*c*) E at the junction.

Ans. 2.70 V; 8.85 pF; 2.70 MV/m

D6.8. Given the volume charge density $\rho_\nu = -2 \times 10^7 \epsilon_0 \sqrt{x}$ C/m^3 in free space, let $V = 0$ at $x = 0$ and let $V = 2$ V at $x = 2.5$ mm. At $x = 1$ mm, find: (*a*) V; (*b*) E_x.

Ans. 0.302 V; −555 V/m

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CHAPTER 6 PROBLEMS

WWW Quizzes

6.1 Consider a coaxial capacitor having inner radius a, outer radius b, unit length, and filled with a material with dielectric constant, ϵ_r. Compare this to a parallel-plate capacitor having plate width w, plate separation d, filled with the same dielectric, and having unit length. Express the ratio b/a in terms of the ratio d/w, such that the two structures will store the same energy for a given applied voltage.

6.2 Let $S = 100$ mm^2, $d = 3$ mm, and $\epsilon_r = 12$ for a parallel-plate capacitor. (a) Calculate the capacitance. (b) After connecting a 6-V battery across the capacitor, calculate E, D, Q, and the total stored electrostatic energy. (c) With the source still connected, the dielectric is carefully withdrawn from between the plates. With the dielectric gone, recalculate E, D, Q, and the energy stored in the capacitor. (d) If the charge and energy found in part (c) are less than the values found in part (b) (which you should have discovered), what became of the missing charge and energy?

6.3 Capacitors tend to be more expensive as their capacitance and maximum voltage $V_{\max}$ increase. The voltage $V_{\max}$ is limited by the field strength at which the dielectric breaks down, E_{BD}. Which of these dielectrics will give the largest $CV_{\max}$ product for equal plate areas? (a) Air: $\epsilon_r = 1$, $E_{BD} = 3$ MV/m. (b) Barium titanate: $\epsilon_r = 1200$, $E_{BD} = 3$ MV/m. (c) Silicon dioxide: $\epsilon_r = 3.78$, $E_{BD} = 16$ MV/m. (d) Polyethylene: $\epsilon_r = 2.26$, $E_{BD} = 4.7$ MV/m.

6.4 An air-filled parallel-plate capacitor with plate separation d and plate area A is connected to a battery that applies a voltage V_0 between plates. With the battery left connected, the plates are moved apart to a distance of $10d$. Determine by what factor each of the following quantities changes: (a) V_0; (b) C; (c) E; (d) D; (e) Q; (f) ρ_S; (g) W_E.

6.5 A parallel-plate capacitor is filled with a nonuniform dielectric characterized by $\epsilon_r = 2 + 2 \times 10^6 x^2$, where x is the distance from one plate in meters. If $S = 0.02$ m^2 and $d = 1$ mm, find C.

6.6 Repeat Problem 6.4, assuming the battery is disconnected before the plate separation is increased.

6.7 Let $\epsilon_{r1} = 2.5$ for $0 < y < 1$ mm, $\epsilon_{r2} = 4$ for $1 < y < 3$ mm, and ϵ_{r3} for $3 < y < 5$ mm (region 3). Conducting surfaces are present at $y = 0$ and

$y = 5$ mm. Calculate the capacitance per square meter of surface area if (*a*) region 3 is air; (*b*) $\epsilon_{r3} = \epsilon_{r1}$; (*c*) $\epsilon_{r3} = \epsilon_{r2}$; (*d*) region 3 is silver.

6.8 A parallel-plate capacitor is made using two circular plates of radius a, with the bottom plate on the xy plane, centered at the origin. The top plate is located at $z = d$, with its center on the z axis. Potential V_0 is on the top plate; the bottom plate is grounded. Dielectric having *radially dependent* permittivity fills the region between plates. The permittivity is given by $\epsilon(\rho) = \epsilon_0(1 + \rho^2/a^2)$. Find (*a*) **E**; (*b*) **D**; (*c*) Q; (*d*) C.

6.9 Two coaxial conducting cylinders of radius 2 cm and 4 cm have a length of 1 m. The region between the cylinders contains a layer of dielectric from $\rho = c$ to $\rho = d$ with $\epsilon_r = 4$. Find the capacitance if (*a*) $c = 2$ cm, $d = 3$ cm; (*b*) $d = 4$ cm, and the volume of the dielectric is the same as in part (*a*).

6.10 A coaxial cable has conductor dimensions of $a = 1.0$ mm and $b = 2.7$ mm. The inner conductor is supported by dielectric spacers ($\epsilon_r = 5$) in the form of washers with a hole radius of 1 mm and an outer radius of 2.7 mm, and with a thickness of 3.0 mm. The spacers are located every 2 cm down the cable. (*a*) By what factor do the spacers increase the capacitance per unit length? (*b*) If 100 V is maintained across the cable, find **E** at all points.

6.11 Two conducting spherical shells have radii $a = 3$ cm and $b = 6$ cm. The interior is a perfect dielectric for which $\epsilon_r = 8$. (*a*) Find C. (*b*) A portion of the dielectric is now removed so that $\epsilon_r = 1.0$, $0 < \phi < \pi/2$, and $\epsilon_r = 8$, $\pi/2 < \phi < 2\pi$. Again find C.

6.12 (*a*) Determine the capacitance of an isolated conducting sphere of radius a in free space (consider an outer conductor existing at $r \to \infty$). (*b*) The sphere is to be covered with a dielectric layer of thickness d and dielectric contant ϵ_r. If $\epsilon_r = 3$, find d in terms of a such that the capacitance is twice that of part (*a*).

6.13 With reference to Figure 6.5, let $b = 6$ m, $h = 15$ m, and the conductor potential be 250 V. Take $\epsilon = \epsilon_0$. Find values for K_1, ρ_L, a, and C.

6.14 Two #16 copper conductors (1.29 mm diameter) are parallel with a separation d between axes. Determine d so that the capacitance between wires in air is 30 pF/m.

6.15 A 2-cm-diameter conductor is suspended in air with its axis 5 cm from a conducting plane. Let the potential of the cylinder be 100 V and that of the plane be 0 V. (*a*) Find the surface charge density on the cylinder at a point nearest the plane. (*b*) Plane at a point nearest the cylinder; (*c*) find the capacitance per unit length.

6.16 Consider an arrangement of two isolated conducting surfaces of any shape that form a capacitor. Use the definitions of capacitance (Eq. (2) in this chapter) and resistance (Eq. (14) in Chapter 5) to show that when the region between the conductors is filled with either conductive material (conductivity σ) or a perfect dielectric (permittivity ϵ), the resulting resistance and capacitance of the structures are related through the simple formula $RC = \epsilon/\sigma$. What basic properties must be true about both the dielectric and the conducting medium for this condition to hold for certain?

6.17 Construct a curvilinear-square map for a coaxial capacitor of 3 cm inner radius and 8 cm outer radius. These dimensions are suitable for the drawing. (*a*) Use your sketch to calculate the capacitance per meter length, assuming $\epsilon_r = 1$. (*b*) Calculate an exact value for the capacitance per unit length.

6.18 Construct a curvilinear-square map of the potential field about two parallel circular cylinders, each of 2.5 cm radius, separated by a center-to-center distance of 13 cm. These dimensions are suitable for the actual sketch if symmetry is considered. As a check, compute the capacitance per meter both from your sketch and from the exact formula. Assume $\epsilon_r = 1$.

6.19 Construct a curvilinear-square map of the potential field between two parallel circular cylinders, one of 4 cm radius inside another of 8 cm radius. The two axes are displaced by 2.5 cm. These dimensions are suitable for the drawing. As a check on the accuracy, compute the capacitance per meter from the sketch and from the exact expression:

$$C = \frac{2\pi\epsilon}{\cosh^{-1}\left[(a^2 + b^2 - D^2)/(2ab)\right]}$$

where a and b are the conductor radii and D is the axis separation.

6.20 A solid conducting cylinder of 4 cm radius is centered within a rectangular conducting cylinder with a 12 cm by 20 cm cross section. (*a*) Make a full-size sketch of one quadrant of this configuration and construct a curvilinear-square map for its interior. (*b*) Assume $\epsilon = \epsilon_0$ and estimate C per meter length.

6.21 The inner conductor of the transmission line shown in Figure 6.13 has a square cross section $2a \times 2a$, whereas the outer square is $4a \times 5a$. The axes are displaced as shown. (*a*) Construct a good-sized drawing of this transmission line, say with $a = 2.5$ cm, and then prepare a curvilinear-square plot of the electrostatic field between the conductors. (*b*) Use the map to calculate the capacitance per meter length if $\epsilon = 1.6\epsilon_0$. (*c*) How would your result to part (*b*) change if $a = 0.6$ cm?

6.22 Two conducting plates, each 3×6 cm, and three slabs of dielectric, each $1 \times 3 \times 6$ cm, and having dielectric constants of 1, 2, and 3, are assembled into a capacitor with $d = 3$ cm. Determine the two values of capacitance obtained by the two possible methods of assembling the capacitor.

6.23 A two-wire transmission line consists of two parallel perfectly conducting cylinders, each having a radius of 0.2 mm, separated by a center-to-center distance of 2 mm. The medium surrounding the wires has $\epsilon_r = 3$ and $\sigma = 1.5$ mS/m. A 100-V battery is connected between the wires. (*a*) Calculate the magnitude of the charge per meter length on each wire. (*b*) Using the result of Problem 6.16, find the battery current.

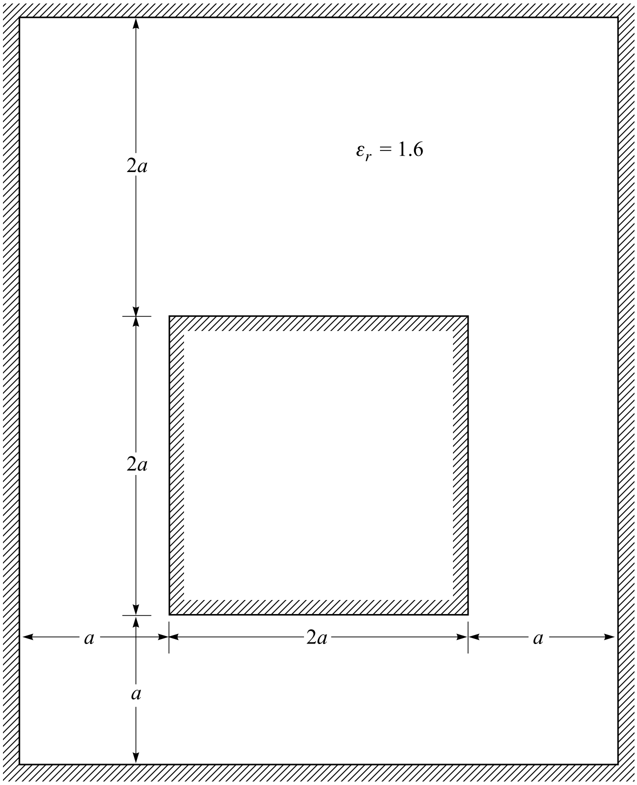

Figure 6.13 See Problem 6.21.

6.24 A potential field in free space is given in spherical coordinates as

$$V(r) = \begin{cases} [\rho_0/(6\epsilon_0)]\,[3a^2 - r^2] & (r \le a) \\ (a^3\rho_0)/(3\epsilon_0 r) & (r \ge a) \end{cases}$$

where ρ_0 and a are constants. (*a*) Use Poisson's equation to find the volume charge density everywhere. (*b*) Find the total charge present.

6.25 Let $V = 2xy^2z^3$ and $\epsilon = \epsilon_0$. Given point $P(1, 2, -1)$, find. (*a*) V at P; (*b*) $\mathbf{E}$ at P; (*c*) ρ_v at P; (*d*) the equation of the equipotential surface passing through P; (*e*) the equation of the streamline passing through P. (*f*) Does V satisfy Laplace's equation?

6.26 Given the spherically symmetric potential field in free space, $V = V_0 e^{-r/a}$, find. (*a*) ρ_v at $r = a$; (*b*) the electric field at $r = a$; (*c*) the total charge.

6.27 Let $V(x, y) = 4e^{2x} + f(x) - 3y^2$ in a region of free space where $\rho_v = 0$. It is known that both E_x and V are zero at the origin. Find $f(x)$ and $V(x, y)$.

6.28 Show that in a homogeneous medium of conductivity σ, the potential field V satisfies Laplace's equation if any volume charge density present does not vary with time.

6.29 Given the potential field $V = (A\rho^4 + B\rho^{-4})\sin 4\phi$: (*a*) Show that $\nabla^2 V = 0$. (*b*) Select A and B so that $V = 100$ V and $|\mathbf{E}| = 500$ V/m at $P(\rho = 1, \phi = 22.5^\circ, z = 2)$.

6.30 A parallel-plate capacitor has plates located at $z = 0$ and $z = d$. The region between plates is filled with a material that contains volume charge of uniform density ρ_0 C/m^3 and has permittivity ϵ. Both plates are held at ground potential. (*a*) Determine the potential field between plates. (*b*) Determine the electric field intensity $\mathbf{E}$ between plates. (*c*) Repeat parts (*a*) and (*b*) for the case of the plate at $z = d$ raised to potential V_0, with the $z = 0$ plate grounded.

6.31 Let $V = (\cos 2\phi)/\rho$ in free space. (*a*) Find the volume charge density at point $A(0.5, 60^\circ, 1)$. (*b*) Find the surface charge density on a conductor surface passing through the point $B(2, 30^\circ, 1)$.

6.32 A uniform volume charge has constant density $\rho_v = \rho_0$ C/m^3 and fills the region $r < a$, in which permittivity ϵ is assumed. A conducting spherical shell is located at $r = a$ and is held at ground potential. Find (*a*) the potential everywhere; (*b*) the electric field intensity, $\mathbf{E}$, everywhere.

6.33 The functions $V_1(\rho, \phi, z)$ and $V_2(\rho, \phi, z)$ both satisfy Laplace's equation in the region $a < \rho < b$, $0 \le \phi < 2\pi$, $-L < z < L$; each is zero on the surfaces $\rho = b$ for $-L < z < L$; $z = -L$ for $a < \rho < b$; and $z = L$ for $a < \rho < b$; and each is 100 V on the surface $\rho = a$ for $-L < z < L$. (*a*) In the region specified, is Laplace's equation satisfied by the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? (*b*) On the boundary surfaces specified, are the potential values given in this problem obtained from the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? (*c*) Are the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$ identical with V_1?

6.34 Consider the parallel-plate capacitor of Problem 6.30, but this time the charged dielectric exists only between $z = 0$ and $z = b$, where $b < d$. Free space fills the region $b < z < d$. Both plates are at ground potential. By solving Laplace's *and* Poisson's equations, find (*a*) $V(z)$ for $0 < z < d$; (*b*) the electric field intensity for $0 < z < d$. No surface charge exists at $z = b$, so both V and $\mathbf{D}$ are continuous there.

6.35 The conducting planes $2x + 3y = 12$ and $2x + 3y = 18$ are at potentials of 100 V and 0, respectively. Let $\epsilon = \epsilon_0$ and find (*a*) V at $P(5, 2, 6)$; (*b*) $\mathbf{E}$ at P.

6.36 The derivation of Laplace's and Poisson's equations assumed constant permittivity, but there are cases of spatially varying permittivity in which the equations will still apply. Consider the vector identity, $\nabla \cdot (\psi \mathbf{G}) = \mathbf{G} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{G}$, where ψ and $\mathbf{G}$ are scalar and vector functions, respectively.

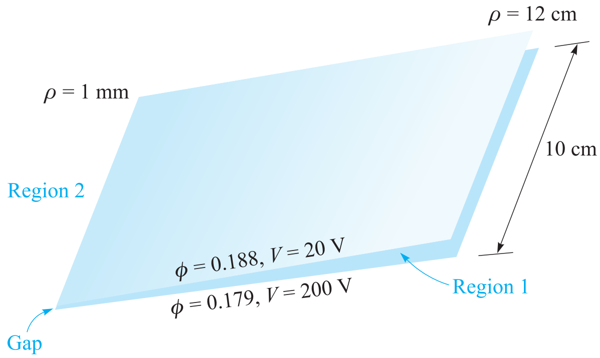

Figure 6.14 See Problem 6.39.

Determine a general rule on the allowed *directions* in which ϵ may vary with respect to the local electric field.

6.37 Coaxial conducting cylinders are located at $\rho = 0.5$ cm and $\rho = 1.2$ cm. The region between the cylinders is filled with a homogeneous perfect dielectric. If the inner cylinder is at 100 V and the outer at 0 V, find (*a*) the location of the 20 V equipotential surface; (*b*) $E_{\rho\,\max}$; (*c*) ϵ_r if the charge per meter length on the inner cylinder is 20 nC/m.

6.38 Repeat Problem 6.37, but with the dielectric only partially filling the volume, within $0 < \phi < \pi$, and with free space in the remaining volume.

6.39 The two conducting planes illustrated in Figure 6.14 are defined by $0.001 < \rho < 0.120$ m, $0 < z < 0.1$ m, $\phi = 0.179$ and 0.188 rad. The medium surrounding the planes is air. For Region 1, $0.179 < \phi < 0.188$; neglect fringing and find (*a*) $V(\phi)$; (*b*) $\mathbf{E}(\rho)$; (*c*) $\mathbf{D}(\rho)$; (*d*) ρ_s on the upper surface of the lower plane; (*e*) Q on the upper surface of the lower plane. (*f*) Repeat parts (*a*) through (*c*) for Region 2 by letting the location of the upper plane be $\phi = .188 - 2\pi$, and then find ρ_s and Q on the lower surface of the lower plane. (*g*) Find the total charge on the lower plane and the capacitance between the planes.

6.40 A parallel-plate capacitor is made using two circular plates of radius a, with the bottom plate on the xy plane, centered at the origin. The top plate is located at $z = d$, with its center on the z axis. Potential V_0 is on the top plate; the bottom plate is grounded. Dielectric having *radially dependent* permittivity fills the region between plates. The permittivity is given by $\epsilon(\rho) = \epsilon_0(1 + \rho^2/a^2)$. Find $(a) V(z)$; $(b)\,\mathbf{E}$; $(c)\,Q$; $(d)\,C$. This is a reprise of Problem 6.8, but it starts with Laplace's equation.

6.41 Concentric conducting spheres are located at $r = 5$ mm and $r = 20$ mm. The region between the spheres is filled with a perfect dielectric. If the inner sphere is at 100 V and the outer sphere is at 0 V (*a*) Find the location of the 20 V equipotential surface. (*b*) Find $E_{r,\max}$. (*c*) Find ϵ_r if the surface charge density on the inner sphere is 1.0 μC/m^2.

6.42 The hemisphere $0 < r < a, 0 < \theta < \pi/2$, is composed of homogeneous conducting material of conductivity σ. The flat side of the hemisphere rests on a perfectly conducting plane. Now, the material within the conical region $0 < \theta < \alpha, 0 < r < a$ is drilled out and replaced with material that is perfectly conducting. An air gap is maintained between the $r = 0$ tip of this new material and the plane. What resistance is measured between the two perfect conductors? Neglect fringing fields.

6.43 Two coaxial conducting cones have their vertices at the origin and the z axis as their axis. Cone A has the point $A(1, 0, 2)$ on its surface, while cone B has the point $B(0, 3, 2)$ on its surface. Let $V_A = 100$ V and $V_B = 20$ V. Find (*a*) α for each cone; (*b*) V at $P(1, 1, 1)$.

6.44 A potential field in free space is given as $V = 100 \ln \tan(\theta/2) + 50$ V. (*a*) Find the maximum value of $|\mathbf{E}_\theta|$ on the surface $\theta = 40^\circ$ for $0.1 < r < 0.8$ m, $60^\circ < \phi < 90^\circ$. (*b*) Describe the surface $V = 80$ V.

6.45 In free space, let $\rho_\nu = 200\epsilon_0/r^{2.4}$. (*a*) Use Poisson's equation to find $V(r)$ if it is assumed that $r^2 E_r \to 0$ when $r \to 0$, and also that $V \to 0$ as $r \to \infty$. (*b*) Now find $V(r)$ by using Gauss's law and a line integral.

6.46 By appropriate solution of Laplace's *and* Poisson's equations, determine the absolute potential at the center of a sphere of radius a, containing uniform volume charge of density ρ_0. Assume permittivity ϵ_0 everywhere. *Hint*: What must be true about the potential and the electric field at $r = 0$ and at $r = a$?

7 CHAPTER

The Steady Magnetic Field

At this point, the concept of a field should be a familiar one. Since we first accepted the experimental law of forces existing between two point charges and defined electric field intensity as the force per unit charge on a test charge in the presence of a second charge, we have discussed numerous fields. These fields possess no real physical basis, for physical measurements must always be in terms of the forces on the charges in the detection equipment. Those charges that are the source cause measurable forces to be exerted on other charges, which we may think of as detector charges. The fact that we attribute a field to the source charges and then determine the effect of this field on the detector charges amounts merely to a division of the basic problem into two parts for convenience.

We will begin our study of the magnetic field with a definition of the magnetic field itself and show how it arises from a current distribution. The effect of this field on other currents, or the second half of the physical problem, will be discussed in Chapter 8. As we did with the electric field, we confine our initial discussion to free-space conditions, and the effect of material media will also be saved for discussion in Chapter 8.

The relation of the steady magnetic field to its source is more complicated than is the relation of the electrostatic field to its source. We will find it necessary to accept several laws temporarily on faith alone. The proof of the laws does exist and is available on the Web site for the disbelievers or the more advanced student. ■

7.1 BIOT-SAVART LAW

The source of the steady magnetic field may be a permanent magnet, an electric field changing linearly with time, or a direct current. We will largely ignore the permanent magnet and save the time-varying electric field for a later discussion. Our present study will concern the magnetic field produced by a differential dc element in free space.

We may think of this differential current element as a vanishingly small section of a current-carrying filamentary conductor, where a filamentary conductor is the limiting

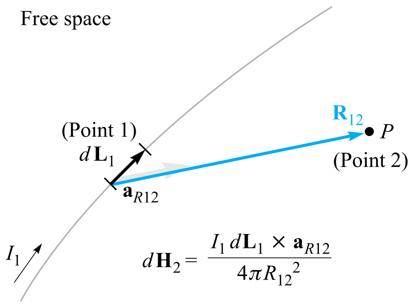

Figure 7.1 The law of Biot-Savart expresses the magnetic field intensity $d\mathbf{H}_2$ produced by a differential current element $I_1 d\mathbf{L}_1$. The direction of $d\mathbf{H}_2$ is into the page.

case of a cylindrical conductor of circular cross section as the radius approaches zero. We assume a current I flowing in a differential vector length of the filament $d\mathbf{L}$. The law of Biot-Savart[1] then states that at any point P the magnitude of the magnetic field intensity produced by the differential element is proportional to the product of the current, the magnitude of the differential length, and the sine of the angle lying between the filament and a line connecting the filament to the point P at which the field is desired; also, the magnitude of the magnetic field intensity is inversely proportional to the square of the distance from the differential element to the point P. The direction of the magnetic field intensity is normal to the plane containing the differential filament and the line drawn from the filament to the point P. Of the two possible normals, that one to be chosen is the one which is in the direction of progress of a right-handed screw turned from $d\mathbf{L}$ through the smaller angle to the line from the filament to P. Using rationalized mks units, the constant of proportionality is $1/4\pi$.

The *Biot-Savart law,* just described in some 150 words, may be written concisely using vector notation as

$$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,d\mathbf{L} \times \mathbf{R}}{4\pi R^3} \tag{1}$$

The units of the *magnetic field intensity* $\mathbf{H}$ are evidently amperes per meter (A/m). The geometry is illustrated in Figure 7.1. Subscripts may be used to indicate the point to which each of the quantities in (1) refers. If we locate the current element at point 1 and describe the point P at which the field is to be determined as point 2, then

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2} \tag{2}$$

[1] Biot and Savart were colleagues of Ampère, and all three were professors of physics at the Collège de France at one time or another. The Biot-Savart law was proposed in 1820.

The law of Biot-Savart is sometimes called *Ampère's law for the current element,* but we will retain the former name because of possible confusion with Ampère's circuital law, to be discussed later.

In some aspects, the Biot-Savart law is reminiscent of Coulomb's law when that law is written for a differential element of charge,

$$d\mathbf{E}_2 = \frac{dQ_1 \mathbf{a}_{R12}}{4\pi\epsilon_0 R_{12}^2}$$

Both show an inverse-square-law dependence on distance, and both show a linear relationship between source and field. The chief difference appears in the direction of the field.

It is impossible to check experimentally the law of Biot-Savart as expressed by (1) or (2) because the differential current element cannot be isolated. We have restricted our attention to direct currents only, so the charge density is not a function of time. The continuity equation in Section 5.2, Eq. (5),

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

therefore shows that

$$\nabla \cdot \mathbf{J} = 0$$

or upon applying the divergence theorem,

$$\oint_s \mathbf{J} \cdot d\mathbf{S} = 0$$

The total current crossing any closed surface is zero, and this condition may be satisfied only by assuming a current flow around a closed path. It is this current flowing in a closed circuit that must be our experimental source, not the differential element.

It follows that only the integral form of the Biot-Savart law can be verified experimentally,

$$\mathbf{H} = \oint \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} \tag{3}$$

Equation (1) or (2), of course, leads directly to the integral form (3), but other differential expressions also yield the same integral formulation. Any term may be added to (1) whose integral around a closed path is zero. That is, any conservative field could be added to (1). The gradient of any scalar field always yields a conservative field, and we could therefore add a term ∇G to (1), where G is a general scalar field, without changing (3) in the slightest. This qualification on (1) or (2) is mentioned to show that if we later ask some foolish questions, not subject to any experimental check, concerning the force exerted by one *differential* current element on another, we should expect foolish answers.

The Biot-Savart law may also be expressed in terms of distributed sources, such as current density **J** and *surface current density* **K**. Surface current flows in a sheet of vanishingly small thickness, and the current density **J**, measured in amperes per square

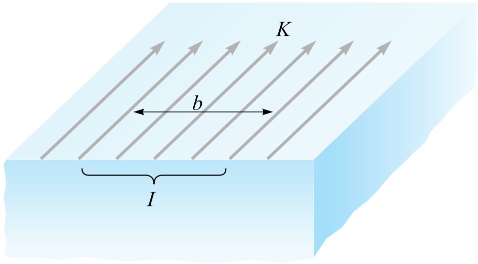

Figure 7.2 The total current I within a transverse width b, in which there is a *uniform* surface current density K, is Kb.

meter, is therefore infinite. Surface current density, however, is measured in amperes per meter width and designated by **K**. If the surface current density is uniform, the total current I in any width b is

$$I = Kb$$

where we assume that the width b is measured perpendicularly to the direction in which the current is flowing. The geometry is illustrated by Figure 7.2. For a nonuniform surface current density, integration is necessary:

$$I = \int K dN \tag{4}$$

where dN is a differential element of the path *across* which the current is flowing. Thus the differential current element $I\,d\mathbf{L}$, where $d\mathbf{L}$ is in the direction of the current, may be expressed in terms of surface current density **K** or current density **J**,

$$I\,d\mathbf{L} = \mathbf{K}\,dS = \mathbf{J}\,dv \tag{5}$$

and alternate forms of the Biot-Savart law obtained,

$$\mathbf{H} = \int_s \frac{\mathbf{K} \times \mathbf{a}_R dS}{4\pi R^2} \tag{6}$$

and

$$\mathbf{H} = \int_{\text{vol}} \frac{\mathbf{J} \times \mathbf{a}_R dv}{4\pi R^2} \tag{7}$$

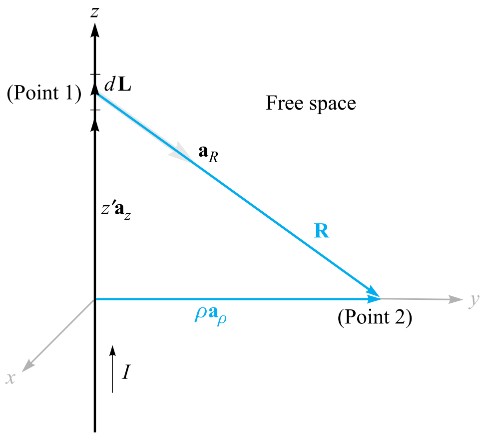

Figure 7.3 An infinitely long straight filament carrying a direct current I. The field at point 2 is $\mathbf{H} = (I/2\pi\rho)\mathbf{a}_\phi$.

We illustrate the application of the Biot-Savart law by considering an infinitely long straight filament. We apply (2) first and then integrate. This, of course, is the same as using the integral form (3) in the first place.[2]

Referring to Figure 7.3, we should recognize the symmetry of this field. No variation with z or with ϕ can exist. Point 2, at which we will determine the field, is therefore chosen in the $z = 0$ plane. The field point $\mathbf{r}$ is therefore $r = \rho\mathbf{a}_\rho$. The source point $\mathbf{r}'$ is given by $\mathbf{r}' = z'\mathbf{a}_z$, and therefore

$$\mathbf{R}_{12} = \mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z'\mathbf{a}_z$$

so that

$$\mathbf{a}_{R12} = \frac{\rho\mathbf{a}_\rho - z'\mathbf{a}_z}{\sqrt{\rho^2 + z'^2}}$$

We take $d\mathbf{L} = dz'\mathbf{a}_z$ and (2) becomes

$$d\mathbf{H}_2 = \frac{I\,dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

Because the current is directed toward increasing values of z', the limits are $-\infty$ and ∞ on the integral, and we have

$$\mathbf{H}_2 = \int_{-\infty}^{\infty} \frac{I\,dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho\,dz'\mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$

[2] The closed path for the current may be considered to include a return filament parallel to the first filament and infinitely far removed. An outer coaxial conductor of infinite radius is another theoretical possibility. Practically, the problem is an impossible one, but we should realize that our answer will be quite accurate near a very long, straight wire having a distant return path for the current.

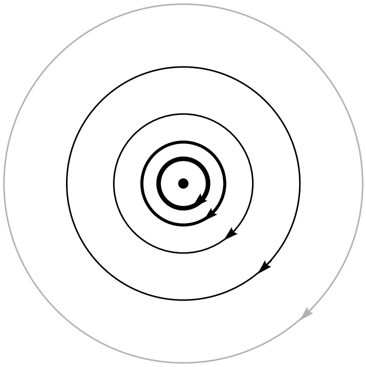

Figure 7.4 The streamlines of the magnetic field intensity about an infinitely long straight filament carrying a direct current *I*. The direction of *I* is into the page.

At this point the unit vector $\mathbf{a}_\phi$ under the integral sign should be investigated, for it is not always a constant, as are the unit vectors of the rectangular coordinate system. A vector is constant when its magnitude and direction are both constant. The unit vector certainly has constant magnitude, but its direction may change. Here $\mathbf{a}_\phi$ changes with the coordinate ϕ but not with ρ or z. Fortunately, the integration here is with respect to z', and $\mathbf{a}_\phi$ is a constant and may be removed from under the integral sign,

Illustrations

$$\mathbf{H}_2 = \frac{I\rho\mathbf{a}_\phi}{4\pi}\int_{-\infty}^{\infty}\frac{dz'}{(\rho^2+z'^2)^{3/2}}$$

$$= \frac{I\rho\mathbf{a}_\phi}{4\pi}\left.\frac{z'}{\rho^2\sqrt{\rho^2+z'^2}}\right|_{-\infty}^{\infty}$$

and

$$\mathbf{H}_2 = \frac{I}{2\pi\rho}\mathbf{a}_\phi \tag{8}$$

The magnitude of the field is not a function of ϕ or z, and it varies inversely with the distance from the filament. The direction of the magnetic-field-intensity vector is circumferential. The streamlines are therefore circles about the filament, and the field may be mapped in cross section as in Figure 7.4.

The separation of the streamlines is proportional to the radius, or inversely proportional to the magnitude of **H**. To be specific, the streamlines have been drawn with curvilinear squares in mind. As yet, we have no name for the family of lines[3] that are perpendicular to these circular streamlines, but the spacing of the streamlines has

[3] If you can't wait, see Section 7.6.

been adjusted so that the addition of this second set of lines will produce an array of curvilinear squares.

A comparison of Figure 7.4 with the map of the *electric* field about an infinite line *charge* shows that the streamlines of the magnetic field correspond exactly to the equipotentials of the electric field, and the unnamed (and undrawn) perpendicular family of lines in the magnetic field corresponds to the streamlines of the electric field. This correspondence is not an accident, but there are several other concepts which must be mastered before the analogy between electric and magnetic fields can be explored more thoroughly.

Using the Biot-Savart law to find **H** is in many respects similar to the use of Coulomb's law to find **E**. Each requires the determination of a moderately complicated integrand containing vector quantities, followed by an integration. When we were concerned with Coulomb's law we solved a number of examples, including the fields of the point charge, line charge, and sheet of charge. The law of Biot-Savart can be used to solve analogous problems in magnetic fields, and some of these problems appear as exercises at the end of the chapter rather than as examples here.

One useful result is the field of the finite-length current element, shown in Figure 7.5. It turns out (see Problem 7.8 at the end of the chapter) that **H** is most easily expressed in terms of the angles α_1 and α_2, as identified in the figure. The result is

$$\mathbf{H} = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi \tag{9}$$

If one or both ends are below point 2, then α_1 is or both α_1 and α_2 are negative.

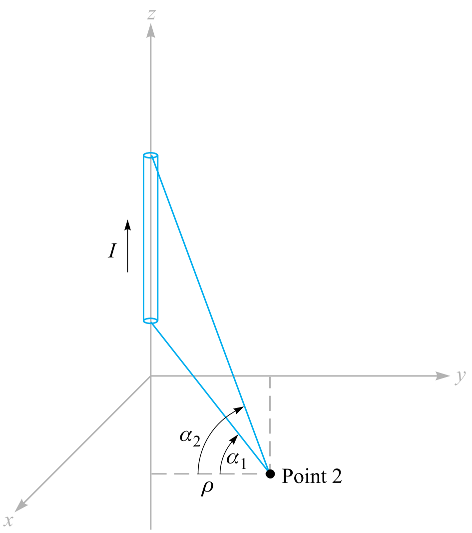

Figure 7.5 The magnetic field intensity caused by a finite-length current filament on the z axis is $(I/4\pi\rho)(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi$.

Equation (9) may be used to find the magnetic field intensity caused by current filaments arranged as a sequence of straight-line segments.

EXAMPLE 7.1

As a numerical example illustrating the use of (9), we determine **H** at $P_2(0.4, 0.3, 0)$ in the field of an 8. A filamentary current is directed inward from infinity to the origin on the positive x axis, and then outward to infinity along the y axis. This arrangement is shown in Figure 7.6.

Solution. We first consider the semi-infinite current on the x axis, identifying the two angles, $\alpha_{1x} = -90^\circ$ and $\alpha_{2x} = \tan^{-1}(0.4/0.3) = 53.1^\circ$. The radial distance ρ is measured from the x axis, and we have $\rho_x = 0.3$. Thus, this contribution to $\mathbf{H}_2$ is

$$\mathbf{H}_{2(x)} = \frac{8}{4\pi(0.3)}(\sin 53.1^\circ + 1)\mathbf{a}_\phi = \frac{2}{0.3\pi}(1.8)\mathbf{a}_\phi = \frac{12}{\pi}\mathbf{a}_\phi$$

The unit vector $\mathbf{a}_\phi$ must also be referred to the x axis. We see that it becomes $-\mathbf{a}_z$. Therefore,

$$\mathbf{H}_{2(x)} = -\frac{12}{\pi}\mathbf{a}_z \text{ A/m}$$

For the current on the y axis, we have $\alpha_{1y} = -\tan^{-1}(0.3/0.4) = -36.9^\circ$, $\alpha_{2y} = 90^\circ$, and $\rho_y = 0.4$. It follows that

$$\mathbf{H}_{2(y)} = \frac{8}{4\pi(0.4)}(1 + \sin 36.9^\circ)(-\mathbf{a}_z) = -\frac{8}{\pi}\mathbf{a}_z \text{ A/m}$$

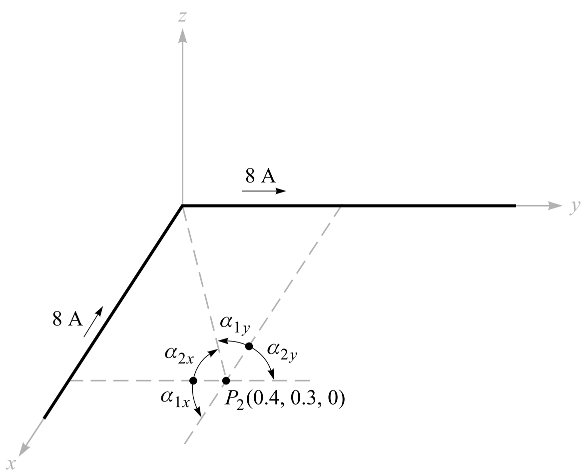

Figure 7.6 The individual fields of two semi-infinite current segments are found by (9) and added to obtain $\mathbf{H}_2$ at P_2.

Adding these results, we have

$$\mathbf{H}_2 = \mathbf{H}_{2(x)} + \mathbf{H}_{2(y)} = -\frac{20}{\pi}\mathbf{a}_z = -6.37\mathbf{a}_z \text{ A/m}$$

D7.1. Given the following values for P_1, P_2, and $I_1 \Delta L_1$, calculate $\Delta\mathbf{H}_2$: (*a*) $P_1(0, 0, 2)$, $P_2(4, 2, 0)$, $2\pi\mathbf{a}_z \mu\text{A}\cdot\text{m}$; (*b*) $P_1(0, 2, 0)$, $P_2(4, 2, 3)$, $2\pi\mathbf{a}_z \mu\text{A}\cdot\text{m}$; (*c*) $P_1(1, 2, 3)$, $P_2(-3, -1, 2)$, $2\pi(-\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z)\mu\text{A}\cdot\text{m}$.

Ans. $-8.51\mathbf{a}_x + 17.01\mathbf{a}_y$ nA/m; $16\mathbf{a}_y$ nA/m; $18.9\mathbf{a}_x - 33.9\mathbf{a}_y + 26.4\mathbf{a}_z$ nA/m

D7.2. A current filament carrying 15 A in the $\mathbf{a}_z$ direction lies along the entire z axis. Find $\mathbf{H}$ in rectangular coordinates at: (*a*) $P_A(\sqrt{20}, 0, 4)$; (*b*) $P_B(2, -4, 4)$.

Ans. $0.534\mathbf{a}_y$ A/m; $0.477\mathbf{a}_x + 0.239\mathbf{a}_y$ A/m

7.2 AMPÈRE'S CIRCUITAL LAW

After solving a number of simple electrostatic problems with Coulomb's law, we found that the same problems could be solved much more easily by using Gauss's law whenever a high degree of symmetry was present. Again, an analogous procedure exists in magnetic fields. Here, the law that helps us solve problems more easily is known as *Ampère's circuital*[4] *law,* sometimes called Ampère's work law. This law may be derived from the Biot-Savart law (see Section 7.7).

Ampère's circuital law states that the line integral of **H** about any *closed* path is exactly equal to the direct current enclosed by that path,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I \tag{10}$$

We define positive current as flowing in the direction of advance of a right-handed screw turned in the direction in which the closed path is traversed.

Referring to Figure 7.7, which shows a circular wire carrying a direct current I, the line integral of **H** about the closed paths lettered a and b results in an answer of I; the integral about the closed path c which passes through the conductor gives an answer less than I and is exactly that portion of the total current that is enclosed by the path c. Although paths a and b give the same answer, the integrands are, of course, different. The line integral directs us to multiply the component of **H** in the direction of the path by a small increment of path length at one point of the path, move along the path to the next incremental length, and repeat the process, continuing until the path is completely traversed. Because **H** will generally vary from point to point, and because paths a and b are not alike, the contributions to the integral made by, say,

[4] The preferred pronunciation puts the accent on "circ-."

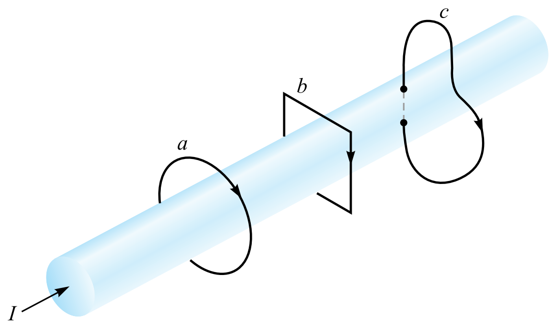

Figure 7.7 A conductor has a total current I. The line integral of $\mathbf{H}$ about the closed paths a and b is equal to I, and the integral around path c is less than I, since the entire current is not enclosed by the path.

each micrometer of path length are quite different. Only the final answers are the same.

We should also consider exactly what is meant by the expression "current enclosed by the path." Suppose we solder a circuit together after passing the conductor once through a rubber band, which we use to represent the closed path. Some strange and formidable paths can be constructed by twisting and knotting the rubber band, but if neither the rubber band nor the conducting circuit is broken, the current enclosed by the path is that carried by the conductor. Now replace the rubber band by a circular ring of spring steel across which is stretched a rubber sheet. The steel loop forms the closed path, and the current-carrying conductor must pierce the rubber sheet if the current is to be enclosed by the path. Again, we may twist the steel loop, and we may also deform the rubber sheet by pushing our fist into it or folding it in any way we wish. A single current-carrying conductor still pierces the sheet once, and this is the true measure of the current enclosed by the path. If we should thread the conductor once through the sheet from front to back and once from back to front, the total current enclosed by the path is the algebraic sum, which is zero.

In more general language, given a closed path, we recognize this path as the perimeter of an infinite number of surfaces (not closed surfaces). Any current-carrying conductor enclosed by the path must pass through every one of these surfaces once. Certainly some of the surfaces may be chosen in such a way that the conductor pierces them twice in one direction and once in the other direction, but the algebraic total current is still the same.

We will find that the nature of the closed path is usually extremely simple and can be drawn on a plane. The simplest surface is, then, that portion of the plane enclosed by the path. We need merely find the total current passing through this region of the plane.

The application of Gauss's law involves finding the total charge enclosed by a closed surface; the application of Ampère's circuital law involves finding the total current enclosed by a closed path.

Let us again find the magnetic field intensity produced by an infinitely long filament carrying a current I. The filament lies on the z axis in free space (as in Figure 7.3), and the current flows in the direction given by $\mathbf{a}_z$. Symmetry inspection comes first, showing that there is no variation with z or ϕ. Next we determine which components of **H** are present by using the Biot-Savart law. Without specifically using the cross product, we may say that the direction of $d\mathbf{H}$ is perpendicular to the plane conaining $d\mathbf{L}$ and **R** and therefore is in the direction of $\mathbf{a}_\phi$. Hence the only component of **H** is H_ϕ, and it is a function only of ρ.

We therefore choose a path, to any section of which **H** is either perpendicular or tangential, and along which H is constant. The first requirement (perpendicularity or tangency) allows us to replace the dot product of Ampère's circuital law with the product of the scalar magnitudes, except along that portion of the path where **H** is normal to the path and the dot product is zero; the second requirement (constancy) then permits us to remove the magnetic field intensity from the integral sign. The integration required is usually trivial and consists of finding the length of that portion of the path to which **H** is parallel.

In our example, the path must be a circle of radius ρ, and Ampère's circuital law becomes

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi\rho = I$$

or

$$H_\phi = \frac{I}{2\pi\rho}$$

as before.

As a second example of the application of Ampère's circuital law, consider an infinitely long coaxial transmission line carrying a uniformly distributed total current I in the center conductor and $-I$ in the outer conductor. The line is shown in Figure 7.8a. Symmetry shows that H is not a function of ϕ or z. In order to determine the components present, we may use the results of the previous example by considering the solid conductors as being composed of a large number of filaments. No filament has a z component of **H**. Furthermore, the H_ρ component at $\phi = 0^\circ$, produced by one filament located at $\rho = \rho_1$, $\phi = \phi_1$, is canceled by the H_ρ component produced by a symmetrically located filament at $\rho = \rho_1$, $\phi = -\phi_1$. This symmetry is illustrated by Figure 7.8b. Again we find only an H_ϕ component which varies with ρ.

A circular path of radius ρ, where ρ is larger than the radius of the inner conductor but less than the inner radius of the outer conductor, then leads immediately to

$$H_\phi = \frac{I}{2\pi\rho} \quad (a < \rho < b)$$

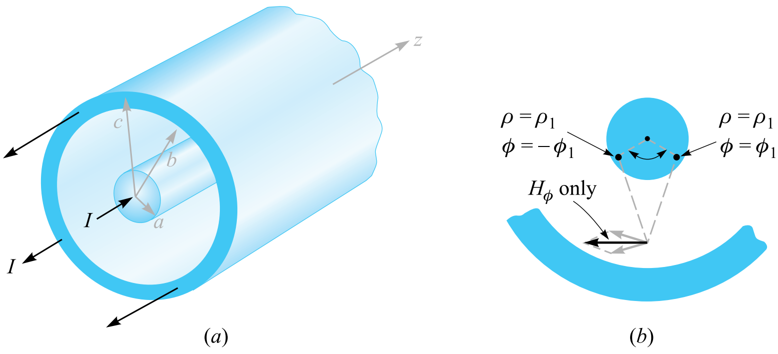

Figure 7.8 (*a*) Cross section of a coaxial cable carrying a uniformly distributed current I in the inner conductor and $-I$ in the outer conductor. The magnetic field at any point is most easily determined by applying Ampère's circuital law about a circular path. (*b*) Current filaments at $\rho = \rho_1, \phi = \pm\phi_1$, produces H_ρ components which cancel. For the total field, $\mathbf{H} = H_\phi \mathbf{a}_\phi$.

If we choose ρ smaller than the radius of the inner conductor, the current enclosed is

$$I_{\text{encl}} = I\frac{\rho^2}{a^2}$$

and

$$2\pi\rho H_\phi = I\frac{\rho^2}{a^2}$$

or

$$H_\phi = \frac{I\rho}{2\pi a^2} \quad (\rho < a)$$

If the radius ρ is larger than the outer radius of the outer conductor, no current is enclosed and

$$H_\phi = 0 \quad (\rho > c)$$

Finally, if the path lies within the outer conductor, we have

$$2\pi\rho H_\phi = I - I\left(\frac{\rho^2 - b^2}{c^2 - b^2}\right)$$

$$H_\phi = \frac{I}{2\pi\rho}\frac{c^2 - \rho^2}{c^2 - b^2} \quad (b < \rho < c)$$

The magnetic-field-strength variation with radius is shown in Figure 7.9 for a coaxial cable in which $b = 3a$, $c = 4a$. It should be noted that the magnetic field intensity **H** is continuous at all the conductor boundaries. In other words, a slight increase in the radius of the closed path does not result in the enclosure of a tremendously different current. The value of H_ϕ shows no sudden jumps.

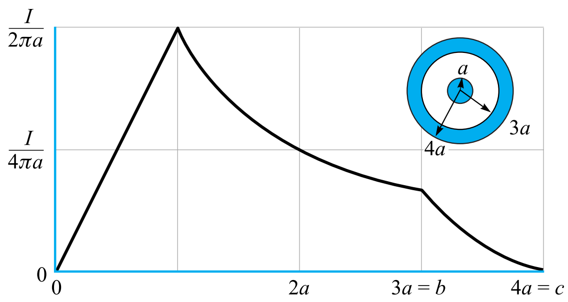

Figure 7.9 The magnetic field intensity as a function of radius in an infinitely long coaxial transmission line with the dimensions shown.

The external field is zero. This, we see, results from equal positive and negative currents enclosed by the path. Each produces an external field of magnitude $I/2\pi\rho$, but complete cancellation occurs. This is another example of "shielding"; such a coaxial cable carrying large currents would, in principle, not produce any noticeable effect in an adjacent circuit.

As a final example, let us consider a sheet of current flowing in the positive y direction and located in the $z = 0$ plane. We may think of the return current as equally divided between two distant sheets on either side of the sheet we are considering. A sheet of uniform surface current density $\mathbf{K} = K_y \mathbf{a}_y$ is shown in Figure 7.10. $\mathbf{H}$ cannot vary with x or y. If the sheet is subdivided into a number of filaments, it is evident that no filament can produce an H_y component. Moreover, the Biot-Savart law shows that the contributions to H_z produced by a symmetrically located pair of filaments cancel. Thus, H_z is zero also; only an H_x component is present. We therefore choose the path 1-1′-2′-2-1 composed of straight-line segments that are either parallel or

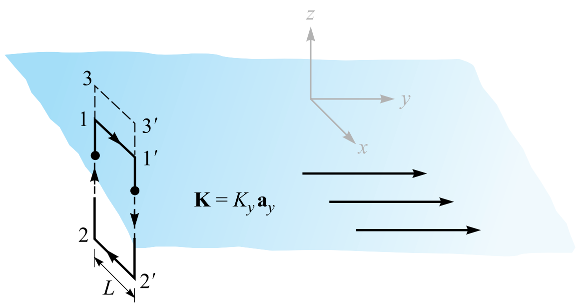

Figure 7.10 A uniform sheet of surface current $\mathbf{K} = K_y \mathbf{a}_y$ in the $z = 0$ plane. $\mathbf{H}$ may be found by applying Ampère's circuital law about the paths 1-1′-2′-2-1 and 3-3′-2′-2-3.

perpendicular to H_x. Ampère's circuital law gives

$$H_{x1}L + H_{x2}(-L) = K_y L$$

or

$$H_{x1} - H_{x2} = K_y$$

If the path 3-3′-2′-2-3 is now chosen, the same current is enclosed, and

$$H_{x3} - H_{x2} = K_y$$

and therefore

$$H_{x3} = H_{x1}$$

It follows that H_x is the same for all positive z. Similarly, H_x is the same for all negative z. Because of the symmetry, then, the magnetic field intensity on one side of the current sheet is the negative of that on the other. Above the sheet,

$$H_x = \tfrac{1}{2}K_y \quad (z > 0)$$

while below it

$$H_x = -\tfrac{1}{2}K_y \quad (z < 0)$$

Letting $\mathbf{a}_N$ be a unit vector normal (outward) to the current sheet, the result may be written in a form correct for all z as

$$\mathbf{H} = \tfrac{1}{2}\mathbf{K} \times \mathbf{a}_N \tag{11}$$

If a second sheet of current flowing in the opposite direction, $\mathbf{K} = -K_y\mathbf{a}_y$, is placed at $z = h$, (11) shows that the field in the region between the current sheets is

$$\mathbf{H} = \mathbf{K} \times \mathbf{a}_N \quad (0 < z < h) \tag{12}$$

and is zero elsewhere,

$$\mathbf{H} = 0 \quad (z < 0, z > h) \tag{13}$$

The most difficult part of the application of Ampère's circuital law is the determination of the components of the field that are present. The surest method is the logical application of the Biot-Savart law and a knowledge of the magnetic fields of simple form.

Problem 7.13 at the end of this chapter outlines the steps involved in applying Ampère's circuital law to an infinitely long solenoid of radius a and uniform current density $K_a\mathbf{a}_\phi$, as shown in Figure 7.11a. For reference, the result is

$$\mathbf{H} = K_a\mathbf{a}_z \quad (\rho < a) \tag{14a}$$

$$\mathbf{H} = 0 \quad (\rho > a) \tag{14b}$$

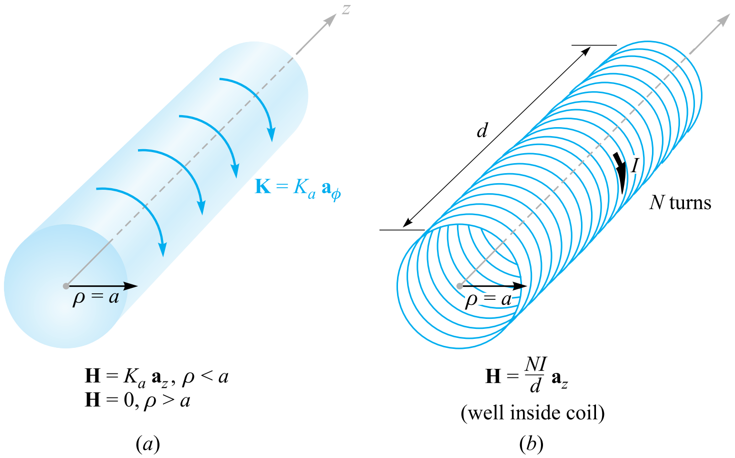

Figure 7.11 (*a*) An ideal solenoid of infinite length with a circular current sheet $\mathbf{K} = K_a\mathbf{a}_\phi$. (*b*) An N-turn solenoid of finite length d.

If the solenoid has a finite length d and consists of N closely wound turns of a filament that carries a current I (Figure 7.11*b*), then the field at points well within the solenoid is given closely by

$$\mathbf{H} = \frac{NI}{d}\mathbf{a}_z \quad \text{(well within the solenoid)} \tag{15}$$

The approximation is useful it if is not applied closer than two radii to the open ends, nor closer to the solenoid surface than twice the separation between turns.

For the toroids shown in Figure 7.12, it can be shown that the magnetic field intensity for the ideal case, Figure 7.12*a*, is

$$\mathbf{H} = K_a\frac{\rho_0 - a}{\rho}\mathbf{a}_\phi \quad \text{(inside toroid)} \tag{16a}$$

$$\mathbf{H} = 0 \quad \text{(outside)} \tag{16b}$$

For the N-turn toroid of Figure 7.12*b*, we have the good approximations,

$$\mathbf{H} = \frac{NI}{2\pi\rho}\mathbf{a}_\phi \quad \text{(inside toroid)} \tag{17a}$$

$$\mathbf{H} = 0 \quad \text{(outside)} \tag{17b}$$

as long as we consider points removed from the toroidal surface by several times the separation between turns.

Toroids having rectangular cross sections are also treated quite readily, as you can see for yourself by trying Problem 7.14.

Accurate formulas for solenoids, toroids, and coils of other shapes are available in Section 2 of the *Standard Handbook for Electrical Engineers* (see References for Chapter 5).

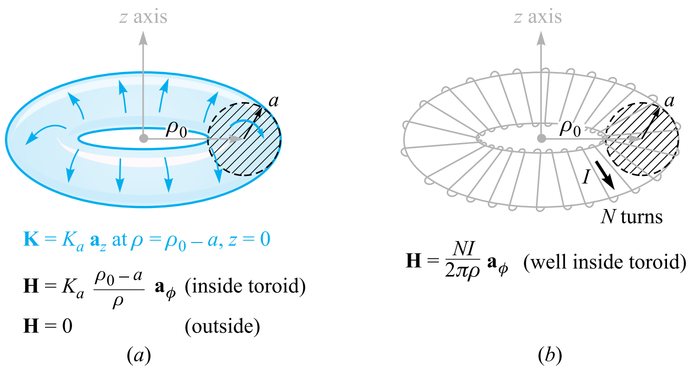

Figure 7.12 (*a*) An ideal toroid carrying a surface current **K** in the direction shown. (*b*) An *N*-turn toroid carrying a filamentary current *I*.

D7.3. Express the value of **H** in rectangular components at $P(0, 0.2, 0)$ in the field of: (*a*) a current filament, 2.5 A in the $\mathbf{a}_z$ direction at $x = 0.1$, $y = 0.3$; (*b*) a coax, centered on the z axis, with $a = 0.3$, $b = 0.5$, $c = 0.6$, $I = 2.5$ A in the $\mathbf{a}_z$ direction in the center conductor; (*c*) three current sheets, $2.7\mathbf{a}_x$ A/m at $y = 0.1$, $-1.4\mathbf{a}_x$ A/m at $y = 0.15$, and $-1.3\mathbf{a}_x$ A/m at $y = 0.25$.

Ans. $1.989\mathbf{a}_x - 1.989\mathbf{a}_y$ A/m; $-0.884\mathbf{a}_x$ A/m; $1.300\mathbf{a}_z$ A/m

7.3 CURL

We completed our study of Gauss's law by applying it to a differential volume element and were led to the concept of divergence. We now apply Ampère's circuital law to the perimeter of a differential surface element and discuss the third and last of the special derivatives of vector analysis, the curl. Our objective is to obtain the point form of Ampère's circuital law.

Again we choose rectangular coordinates, and an incremental closed path of sides Δx and Δy is selected (Figure 7.13). We assume that some current, as yet unspecified, produces a reference value for **H** at the *center* of this small rectangle,

$$\mathbf{H}_0 = H_{x0}\mathbf{a}_x + H_{y0}\mathbf{a}_y + H_{z0}\mathbf{a}_z$$

The closed line integral of **H** about this path is then approximately the sum of the four values of $\mathbf{H} \cdot \Delta\mathbf{L}$ on each side. We choose the direction of traverse as 1-2-3-4-1, which corresponds to a current in the $\mathbf{a}_z$ direction, and the first contribution is therefore

$$(\mathbf{H} \cdot \Delta\mathbf{L})_{1-2} = H_{y,1-2}\Delta y$$

The value of H_y *on this section* of the path may be given in terms of the reference value H_{y0} at the center of the rectangle, the rate of change of H_y with x, and the

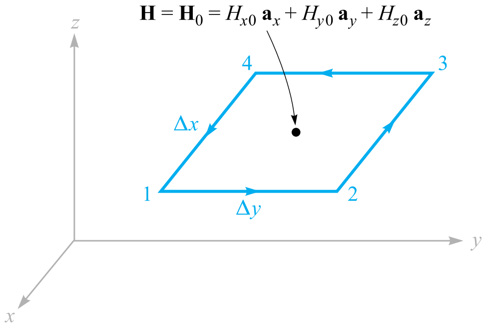

Figure 7.13 An incremental closed path in rectangular coordinates is selected for the application of Ampère's circuital law to determine the spatial rate of change of **H**.

distance $\Delta x/2$ from the center to the midpoint of side 1–2:

$$H_{y,1-2} \doteq H_{y0} + \frac{\partial H_y}{\partial x}\left(\frac{1}{2}\Delta x\right)$$

Thus

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{1-2} \doteq \left(H_{y0} + \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)\Delta y$$

Along the next section of the path we have

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{2-3} \doteq H_{x,2-3}(-\Delta x) \doteq -\left(H_{x0} + \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y\right)\Delta x$$

Continuing for the remaining two segments and adding the results,

$$\oint \mathbf{H}\cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x\,\Delta y$$

By Ampère's circuital law, this result must be equal to the current enclosed by the path, or the current crossing any surface bounded by the path. If we assume a general current density **J**, the enclosed current is then $\Delta I \doteq J_z \Delta x \Delta y$, and

$$\oint \mathbf{H}\cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x\,\Delta y \doteq J_z \Delta x\,\Delta y$$

or

$$\frac{\oint \mathbf{H}\cdot d\mathbf{L}}{\Delta x\,\Delta y} \doteq \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \doteq J_z$$

As we cause the closed path to shrink, the preceding expression becomes more nearly exact, and in the limit we have the equality

$$\lim_{\Delta x,\Delta y\to 0} \frac{\oint \mathbf{H}\cdot d\mathbf{L}}{\Delta x\,\Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \qquad (18)$$

After beginning with Ampère's circuital law equating the closed line integral of **H** to the current enclosed, we have now arrived at a relationship involving the closed line integral of **H** *per unit area* enclosed and the current *per unit area* enclosed, or current density. We performed a similar analysis in passing from the integral form of Gauss's law, involving flux through a closed surface and charge enclosed, to the point form, relating flux through a closed surface *per unit volume* enclosed and charge *per unit volume* enclosed, or volume charge density. In each case a limit is necessary to produce an equality.

If we choose closed paths that are oriented perpendicularly to each of the remaining two coordinate axes, analogous processes lead to expressions for the x and y components of the current density,

$$\lim_{\Delta y, \Delta z \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \tag{19}$$

and

$$\lim_{\Delta z, \Delta x \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \tag{20}$$

Comparing (18)–(20), we see that a component of the current density is given by the limit of the quotient of the closed line integral of **H** about a small path in a plane normal to that component and of the area enclosed as the path shrinks to zero. This limit has its counterpart in other fields of science and long ago received the name of *curl*. The curl of any vector is a vector, and any component of the curl is given by the limit of the quotient of the closed line integral of the vector about a small path in a plane normal to that component desired and the area enclosed, as the path shrinks to zero. It should be noted that this definition of curl does not refer specifically to a particular coordinate system. The mathematical form of the definition is

$$(\text{curl } \mathbf{H})_N = \lim_{\Delta S_N \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta S_N} \tag{21}$$

where ΔS_N is the planar area enclosed by the closed line integral. The N subscript indicates that the component of the curl is that component which is *normal* to the surface enclosed by the closed path. It may represent any component in any coordinate system.

In rectangular coordinates, the definition (21) shows that the x, y, and z components of the curl **H** are given by (18)–(20), and therefore

$$\text{curl } \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \mathbf{a}_z \tag{22}$$

This result may be written in the form of a determinant,

$$\text{curl } \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \tag{23}$$

and may also be written in terms of the vector operator,

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} \tag{24}$$

Equation (22) is the result of applying the definition (21) to the rectangular coordinate system. We obtained the z component of this expression by evaluating Ampère's circuital law about an incremental path of sides Δx and Δy, and we could have obtained the other two components just as easily by choosing the appropriate paths. Equation (23) is a neat method of storing the rectangular coordinate expression for curl; the form is symmetrical and easily remembered. Equation (24) is even more concise and leads to (22) upon applying the definitions of the cross product and vector operator.

The expressions for curl **H** in cylindrical and spherical coordinates are derived in Appendix A by applying the definition (21). Although they may be written in determinant form, as explained there, the determinants do not have one row of unit vectors on top and one row of components on the bottom, and they are not easily memorized. For this reason, the curl expansions in cylindrical and spherical coordinates that follow here and appear inside the back cover are usually referred to whenever necessary.

$$\nabla \times \mathbf{H} = \left(\frac{1}{\rho} \frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} \right) \mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} \right) \mathbf{a}_\phi + \left(\frac{1}{\rho} \frac{\partial (\rho H_\phi)}{\partial \rho} - \frac{1}{\rho} \frac{\partial H_\rho}{\partial \phi} \right) \mathbf{a}_z \quad \text{(cylindrical)} \tag{25}$$

$$\nabla \times \mathbf{H} = \frac{1}{r \sin\theta} \left(\frac{\partial (H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi} \right) \mathbf{a}_r + \frac{1}{r} \left(\frac{1}{\sin\theta} \frac{\partial H_r}{\partial \phi} - \frac{\partial (r H_\phi)}{\partial r} \right) \mathbf{a}_\theta + \frac{1}{r} \left(\frac{\partial (r H_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta} \right) \mathbf{a}_\phi \quad \text{(spherical)} \tag{26}$$

Although we have described curl as a line integral per unit area, this does not provide everyone with a satisfactory physical picture of the nature of the curl operation, for the closed line integral itself requires physical interpretation. This integral was first met in the electrostatic field, where we saw that $\oint \mathbf{E} \cdot d\mathbf{L} = 0$. Inasmuch as the integral was zero, we did not belabor the physical picture. More recently we have discussed the closed line integral of **H**, $\oint \mathbf{H} \cdot d\mathbf{L} = I$. Either of these closed line integrals is also known by the name of *circulation,* a term borrowed from the field of fluid dynamics.

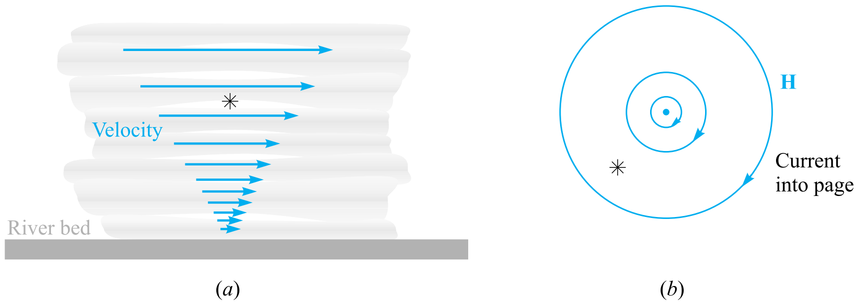

Figure 7.14 (*a*) The curl meter shows a component of the curl of the water velocity into the page. (*b*) The curl of the magnetic field intensity about an infinitely long filament is shown.

The circulation of **H**, or $\oint \mathbf{H} \cdot d\mathbf{L}$, is obtained by multiplying the component of **H** parallel to the specified closed path at each point along it by the differential path length and summing the results as the differential lengths approach zero and as their number becomes infinite. We do not require a vanishingly small path. Ampère's circuital law tells us that if **H** does possess circulation about a given path, then current passes through this path. In electrostatics we see that the circulation of **E** is zero about every path, a direct consequence of the fact that zero work is required to carry a charge around a closed path.

We may describe curl as *circulation per unit area*. The closed path is vanishingly small, and curl is defined at a point. The curl of **E** must be zero, for the circulation is zero. The curl of **H** is not zero, however; the circulation of **H** per unit area is the current density by Ampère's circuital law [or (18), (19), and (20)].

Skilling[5] suggests the use of a very small paddle wheel as a "curl meter." Our vector quantity, then, must be thought of as capable of applying a force to each blade of the paddle wheel, the force being proportional to the component of the field normal to the surface of that blade. To test a field for curl, we dip our paddle wheel into the field, with the axis of the paddle wheel lined up with the direction of the component of curl desired, and note the action of the field on the paddle. No rotation means no curl; larger angular velocities mean greater values of the curl; a reversal in the direction of spin means a reversal in the sign of the curl. To find the direction of the vector curl and not merely to establish the presence of any particular component, we should place our paddle wheel in the field and hunt around for the orientation which produces the greatest torque. The direction of the curl is then along the axis of the paddle wheel, as given by the right-hand rule.

As an example, consider the flow of water in a river. Figure 7.14*a* shows the longitudinal section of a wide river taken at the middle of the river. The water velocity is zero at the bottom and increases linearly as the surface is approached. A paddle wheel placed in the position shown, with its axis perpendicular to the paper, will turn in a clockwise direction, showing the presence of a component of curl in the direction

[5] See the References at the end of the chapter.

of an inward normal to the surface of the page. If the velocity of water does not change as we go up- or downstream and also shows no variation as we go across the river (or even if it decreases in the same fashion toward either bank), then this component is the only component present at the center of the stream, and the curl of the water velocity has a direction into the page.

In Figure 7.14*b*, the streamlines of the magnetic field intensity about an infinitely long filamentary conductor are shown. The curl meter placed in this field of curved lines shows that a larger number of blades have a clockwise force exerted on them but that this force is in general smaller than the counterclockwise force exerted on the smaller number of blades closer to the wire. It seems possible that if the curvature of the streamlines is correct and also if the variation of the field strength is just right, the net torque on the paddle wheel may be zero. Actually, the paddle wheel does not rotate in this case, for since $\mathbf{H} = (I/2\pi\rho)\mathbf{a}_\phi$, we may substitute into (25) obtaining

$$\text{curl } \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho}\mathbf{a}_z = 0$$

EXAMPLE 7.2

As an example of the evaluation of curl **H** from the definition and of the evaluation of another line integral, suppose that $\mathbf{H} = 0.2z^2\mathbf{a}_x$ for $z > 0$, and $\mathbf{H} = 0$ elsewhere, as shown in Figure 7.15. Calculate $\oint \mathbf{H} \cdot d\mathbf{L}$ about a square path with side d, centered at $(0, 0, z_1)$ in the $y = 0$ plane where $z_1 > d/2$.

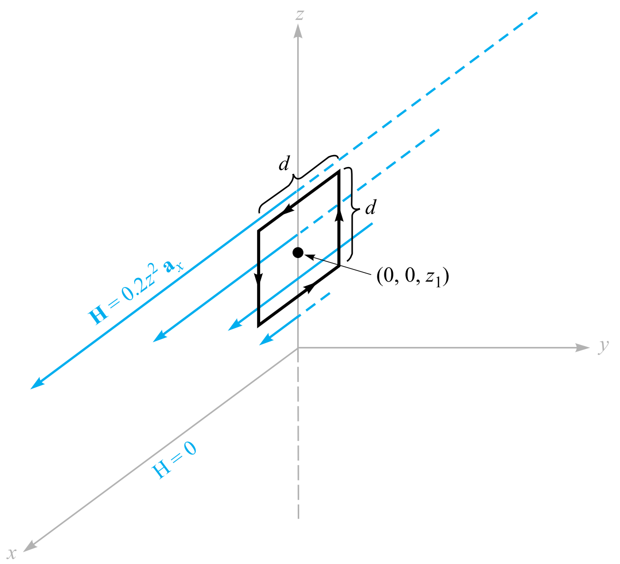

Figure 7.15 A square path of side d with its center on the z axis at $z = z_1$ is used to evaluate $\oint \mathbf{H} \cdot d\mathbf{L}$ and find curl **H**.

Solution. We evaluate the line integral of $\mathbf{H}$ along the four segments, beginning at the top:

$$\oint \mathbf{H}\cdot d\mathbf{L} = 0.2\left(z_1 + \tfrac{1}{2}d\right)^2 d + 0 - 0.2\left(z_1 - \tfrac{1}{2}d\right)^2 d + 0$$
$$= 0.4 z_1 d^2$$

In the limit as the area approaches zero, we find

$$(\nabla \times \mathbf{H})_y = \lim_{d\to 0} \frac{\oint \mathbf{H}\cdot d\mathbf{L}}{d^2} = \lim_{d\to 0} \frac{0.4 z_1 d^2}{d^2} = 0.4 z_1$$

The other components are zero, so $\nabla \times \mathbf{H} = 0.4 z_1 \mathbf{a}_y$.

To evaluate the curl without trying to illustrate the definition or the evaluation of a line integral, we simply take the partial derivative indicated by (23):

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0.2z^2 & 0 & 0 \end{vmatrix} = \frac{\partial}{\partial z}(0.2z^2)\mathbf{a}_y = 0.4z\mathbf{a}_y$$

which checks with the preceding result when $z = z_1$.

Returning now to complete our original examination of the application of Ampère's circuital law to a differential-sized path, we may combine (18)–(20), (22), and (24),

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z = \mathbf{J} \tag{27}$$

and write the *point form of Ampère's circuital law,*

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}} \tag{28}$$

This is the second of Maxwell's four equations as they apply to non-time-varying conditions. We may also write the third of these equations at this time; it is the point form of $\oint \mathbf{E}\cdot d\mathbf{L} = 0$, or

$$\boxed{\nabla \times \mathbf{E} = 0} \tag{29}$$

The fourth equation appears in Section 7.5.

D7.4. (*a*) Evaluate the closed line integral of **H** about the rectangular path $P_1(2, 3, 4)$ to $P_2(4, 3, 4)$ to $P_3(4, 3, 1)$ to $P_4(2, 3, 1)$ to P_1, given $\mathbf{H} = 3z\mathbf{a}_x - 2x^3\mathbf{a}_z$ A/m. (*b*) Determine the quotient of the closed line integral and the area enclosed by the path as an approximation to $(\nabla \times \mathbf{H})_y$. (*c*) Determine $(\nabla \times \mathbf{H})_y$ at the center of the area.

Ans. 354 A; 59 A/m^2; 57 A/m^2

D7.5. Calculate the value of the vector current density: (*a*) in rectangular coordinates at $P_A(2, 3, 4)$ if $\mathbf{H} = x^2z\mathbf{a}_y - y^2x\mathbf{a}_z$; (*b*) in cylindrical coordinates at $P_B(1.5, 90°, 0.5)$ if $\mathbf{H} = \frac{2}{\rho}(\cos 0.2\phi)\mathbf{a}_\rho$; (*c*) in spherical coordinates at $P_C(2, 30°, 20°)$ if $\mathbf{H} = \frac{1}{\sin\theta}\mathbf{a}_\theta$.

Ans. $-16\mathbf{a}_x + 9\mathbf{a}_y + 16\mathbf{a}_z$ A/m^2; $0.055\mathbf{a}_z$ A/m^2; $\mathbf{a}_\phi$ A/m^2

7.4 STOKES' THEOREM

Although Section 7.3 was devoted primarily to a discussion of the curl operation, the contribution to the subject of magnetic fields should not be overlooked. From Ampère's circuital law we derived one of Maxwell's equations, $\nabla \times \mathbf{H} = \mathbf{J}$. This latter equation should be considered the point form of Ampère's circuital law and applies on a "per-unit-area" basis. In this section we shall again devote a major share of the material to the mathematical theorem known as Stokes' theorem, but in the process we will show that we may obtain Ampère's circuital law from $\nabla \times \mathbf{H} = \mathbf{J}$. In other words, we are then prepared to obtain the integral form from the point form or to obtain the point form from the integral form.

Consider the surface S of Figure 7.16, which is broken up into incremental surfaces of area ΔS. If we apply the definition of the curl to one of these incremental surfaces, then

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}{\Delta S} \doteq (\nabla \times \mathbf{H})_N$$

where the N subscript again indicates the right-hand normal to the surface. The subscript on $d\mathbf{L}_{\Delta S}$ indicates that the closed path is the perimeter of an incremental area ΔS. This result may also be written

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N$$

or

$$\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N \Delta S = (\nabla \times \mathbf{H}) \cdot \Delta\mathbf{S}$$

where $\mathbf{a}_N$ is a unit vector in the direction of the right-hand normal to ΔS.

Now let us determine this circulation for every ΔS comprising S and sum the results. As we evaluate the closed line integral for each ΔS, some cancellation will occur

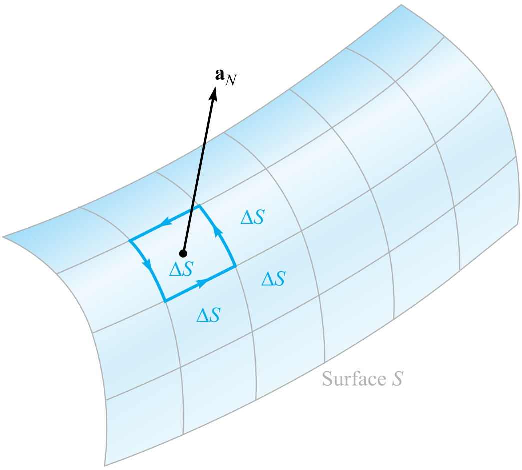

Figure 7.16 The sum of the closed line integrals about the perimeter of every ΔS is the same as the closed line integral about the perimeter of S because of cancellation on every interior path.

because every *interior* wall is covered once in each direction. The only boundaries on which cancellation cannot occur form the outside boundary, the path enclosing S. Therefore we have

$$\oint \mathbf{H} \cdot d\mathbf{L} \equiv \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} \tag{30}$$

where $d\mathbf{L}$ is taken only on the perimeter of S.

Equation (30) is an identity, holding for any vector field, and is known as *Stokes' theorem*.

EXAMPLE 7.3

A numerical example may help to illustrate the geometry involved in Stokes' theorem. Consider the portion of a sphere shown in Figure 7.17. The surface is specified by $r = 4, 0 \le \theta \le 0.1\pi, 0 \le \phi \le 0.3\pi$, and the closed path forming its perimeter is composed of three circular arcs. We are given the field $\mathbf{H} = 6r \sin\phi \mathbf{a}_r + 18r \sin\theta \cos\phi \mathbf{a}_\phi$ and are asked to evaluate each side of Stokes' theorem.

Solution. The first path segment is described in spherical coordinates by $r = 4, 0 \le \theta \le 0.1\pi, \phi = 0$; the second one by $r = 4, \theta = 0.1\pi, 0 \le \phi \le 0.3\pi$; and the third by $r = 4, 0 \le \theta \le 0.1\pi, \phi = 0.3\pi$. The differential path element $d\mathbf{L}$ is the vector sum of the three differential lengths of the spherical coordinate system first discussed in Section 1.9,

$$d\mathbf{L} = dr\, \mathbf{a}_r + r\, d\theta\, \mathbf{a}_\theta + r \sin\theta\, d\phi\, \mathbf{a}_\phi$$

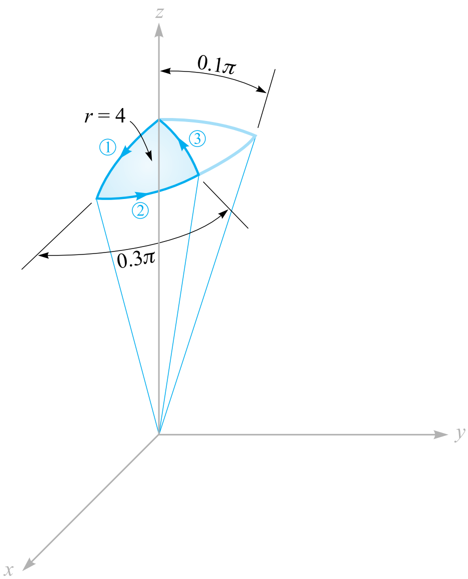

Figure 7.17 A portion of a spherical cap is used as a surface and a closed path to illustrate Stokes' theorem.

The first term is zero on all three segments of the path since $r = 4$ and $dr = 0$, the second is zero on segment 2 as θ is constant, and the third term is zero on both segments 1 and 3. Thus,

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_1 H_\theta r \, d\theta + \int_2 H_\phi r \sin\theta \, d\phi + \int_3 H_\theta r \, d\theta$$

Because $H_\theta = 0$, we have only the second integral to evaluate,

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{0.3\pi} [18(4) \sin 0.1\pi \cos\phi] 4 \sin 0.1\pi \, d\phi$$
$$= 288 \sin^2 0.1\pi \sin 0.3\pi = 22.2 \text{ A}$$

We next attack the surface integral. First, we use (26) to find

$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}(36r\sin\theta\cos\theta\cos\phi)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta} 6r\cos\phi - 36r\sin\theta\cos\phi\right)\mathbf{a}_\theta$$

Because $d\mathbf{S} = r^2 \sin\theta \, d\theta \, d\phi \, \mathbf{a}_r$, the integral is

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_0^{0.3\pi} \int_0^{0.1\pi} (36 \cos\theta \cos\phi) 16 \sin\theta \, d\theta \, d\phi$$
$$= \int_0^{0.3\pi} 576 \left(\tfrac{1}{2} \sin^2\theta\right) \Big|_0^{0.1\pi} \cos\phi \, d\phi$$
$$= 288 \sin^2 0.1\pi \sin 0.3\pi = 22.2 \text{ A}$$

Thus, the results check Stokes' theorem, and we note in passing that a current of 22.2 A is flowing upward through this section of a spherical cap.

Next, let us see how easy it is to obtain Ampère's circuital law from $\nabla \times \mathbf{H} = \mathbf{J}$. We merely have to dot each side by $d\mathbf{S}$, integrate each side over the same (open) surface S, and apply Stokes' theorem:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} = \oint \mathbf{H} \cdot d\mathbf{L}$$

The integral of the current density over the surface S is the total current I passing through the surface, and therefore

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

This short derivation shows clearly that the current I, described as being "enclosed by the closed path," is also the current passing through any of the infinite number of surfaces that have the closed path as a perimeter.

Stokes' theorem relates a surface integral to a closed line integral. It should be recalled that the divergence theorem relates a volume integral to a closed surface integral. Both theorems find their greatest use in general vector proofs. As an example, let us find another expression for $\nabla \cdot \nabla \times \mathbf{A}$, where $\mathbf{A}$ represents any vector field. The result must be a scalar (why?), and we may let this scalar be T, or

$$\nabla \cdot \nabla \times \mathbf{A} = T$$

Multiplying by dv and integrating throughout any volume v,

$$\int_{\text{vol}} (\nabla \cdot \nabla \times \mathbf{A})\, dv = \int_{\text{vol}} T\, dv$$

we first apply the divergence theorem to the left side, obtaining

$$\oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_{\text{vol}} T\, dv$$

The left side is the surface integral of the curl of $\mathbf{A}$ over the *closed* surface surrounding the volume v. Stokes' theorem relates the surface integral of the curl of $\mathbf{A}$ over the *open* surface enclosed by a given closed path. If we think of the path as the opening of a laundry bag and the open surface as the surface of the bag itself, we see that as we gradually approach a closed surface by pulling on the drawstrings, the closed path becomes smaller and smaller and finally disappears as the surface becomes closed. Hence, the application of Stokes' theorem to a *closed* surface produces a zero result, and we have

$$\int_{\text{vol}} T\, dv = 0$$

Because this is true for any volume, it is true for the differential volume dv,

$$T\,dv = 0$$

and therefore

$$T = 0$$

or

$$\boxed{\nabla \cdot \nabla \times \mathbf{A} \equiv 0} \tag{31}$$

Equation (31) is a useful identity of vector calculus.[6] Of course, it may also be proven easily by direct expansion in rectangular coordinates.

Let us apply the identity to the non-time-varying magnetic field for which

$$\nabla \times \mathbf{H} = \mathbf{J}$$

This shows quickly that

$$\nabla \cdot \mathbf{J} = 0$$

which is the same result we obtained earlier in the chapter by using the continuity equation.

Before introducing several new magnetic field quantities in the following section, we may review our accomplishments at this point. We initially accepted the Biot-Savart law as an experimental result,

$$\mathbf{H} = \oint \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

and tentatively accepted Ampère's circuital law, subject to later proof,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

From Ampère's circuital law the definition of curl led to the point form of this same law,

$$\nabla \times \mathbf{H} = \mathbf{J}$$

We now see that Stokes' theorem enables us to obtain the integral form of Ampère's circuital law from the point form.

D7.6. Evaluate both sides of Stokes' theorem for the field $\mathbf{H} = 6xy\mathbf{a}_x - 3y^2\mathbf{a}_y$ A/m and the rectangular path around the region, $2 \le x \le 5$, $-1 \le y \le 1$, $z = 0$. Let the positive direction of $d\mathbf{S}$ be $\mathbf{a}_z$.

Ans. -126 A; -126 A

[6] This and other vector identities are tabulated in Appendix A.3.

7.5 MAGNETIC FLUX AND MAGNETIC FLUX DENSITY

In free space, let us define the *magnetic flux density* **B** as

$$\boxed{\mathbf{B} = \mu_0 \mathbf{H}} \quad \text{(free space only)} \tag{32}$$

where **B** is measured in webers per square meter (Wb/m^2) or in a newer unit adopted in the International System of Units, tesla (T). An older unit that is often used for magnetic flux density is the gauss (G), where 1 T or 1Wb/m^2 is the same as 10, 000 G. The constant μ_0 is not dimensionless and has the *defined value* for free space, in henrys per meter (H/m), of

$$\boxed{\mu_0 = 4\pi \times 10^{-7} \text{ H/m}} \tag{33}$$

The name given to μ_0 is the *permeability* of free space.

We should note that since **H** is measured in amperes per meter, the weber is dimensionally equal to the product of henrys and amperes. Considering the henry as a new unit, the weber is merely a convenient abbreviation for the product of henrys and amperes. When time-varying fields are introduced, it will be shown that a weber is also equivalent to the product of volts and seconds.

The magnetic-flux-density vector **B**, as the name weber per square meter implies, is a member of the flux-density family of vector fields. One of the possible analogies between electric and magnetic fields[7] compares the laws of Biot-Savart and Coulomb, thus establishing an analogy between **H** and **E**. The relations $\mathbf{B} = \mu_0 \mathbf{H}$ and $\mathbf{D} = \epsilon_0 \mathbf{E}$ then lead to an analogy between **B** and **D**. If **B** is measured in teslas or webers per square meter, then magnetic flux should be measured in webers. Let us represent magnetic flux by Φ and define Φ as the flux passing through any designated area,

$$\boxed{\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} \text{ Wb}} \tag{34}$$

Our analogy should now remind us of the electric flux Ψ, measured in coulombs, and of Gauss's law, which states that the total flux passing through any closed surface is equal to the charge enclosed,

$$\Psi = \oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

The charge Q is the source of the lines of electric flux and these lines begin and terminate on positive and negative charges, respectively.

[7] An alternate analogy is presented in Section 9.2.

No such source has ever been discovered for the lines of magnetic flux. In the example of the infinitely long straight filament carrying a direct current I, the **H** field formed concentric circles about the filament. Because $\mathbf{B} = \mu_0 \mathbf{H}$, the **B** field is of the same form. The magnetic flux lines are closed and do not terminate on a "magnetic charge." For this reason Gauss's law for the magnetic field is

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{35}$$

and application of the divergence theorem shows us that

$$\nabla \cdot \mathbf{B} = 0 \tag{36}$$

Equation (36) is the last of Maxwell's four equations as they apply to static electric fields and steady magnetic fields. Collecting these equations, we then have for static electric fields and steady magnetic fields

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_v \\ \nabla \times \mathbf{E} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned} \tag{37}$$

To these equations we may add the two expressions relating **D** to **E** and **B** to **H** in free space,

$$\mathbf{D} = \epsilon_0 \mathbf{E} \tag{38}$$

$$\mathbf{B} = \mu_0 \mathbf{H} \tag{39}$$

We have also found it helpful to define an electrostatic potential,

$$\mathbf{E} = -\nabla V \tag{40}$$

and we will discuss a potential for the steady magnetic field in the following section. In addition, we extended our coverage of electric fields to include conducting materials and dielectrics, and we introduced the polarization **P.** A similar treatment will be applied to magnetic fields in the next chapter.

Returning to (37), it may be noted that these four equations specify the divergence and curl of an electric and a magnetic field. The corresponding set of four integral equations that apply to static electric fields and steady magnetic fields is

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = Q = \int_{vol} \rho_\nu d\nu$$
$$\oint \mathbf{E}\cdot d\mathbf{L} = 0$$
$$\oint \mathbf{H}\cdot d\mathbf{L} = I = \int_S \mathbf{J}\cdot d\mathbf{S} \tag{41}$$
$$\oint_S \mathbf{B}\cdot d\mathbf{S} = 0$$

Our study of electric and magnetic fields would have been much simpler if we could have begun with either set of equations, (37) or (41). With a good knowledge of vector analysis, such as we should now have, either set may be readily obtained from the other by applying the divergence theorem or Stokes' theorem. The various experimental laws can be obtained easily from these equations.

As an example of the use of flux and flux density in magnetic fields, let us find the flux between the conductors of the coaxial line of Figure 7.8*a*. The magnetic field intensity was found to be

$$H_\phi = \frac{I}{2\pi\rho} \qquad (a < \rho < b)$$

and therefore

$$\mathbf{B} = \mu_0\mathbf{H} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi$$

The magnetic flux contained between the conductors in a length d is the flux crossing any radial plane extending from $\rho = a$ to $\rho = b$ and from, say, $z = 0$ to $z = d$

$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi \cdot d\rho\, dz\, \mathbf{a}_\phi$$

or

$$\Phi = \frac{\mu_0 I d}{2\pi}\ln\frac{b}{a} \tag{42}$$

This expression will be used later to obtain the inductance of the coaxial transmission line.

D7.7. A solid conductor of circular cross section is made of a homogeneous nonmagnetic material. If the radius $a = 1$ mm, the conductor axis lies on the z axis, and the total current in the $\mathbf{a}_z$ direction is 20 A, find: (*a*) H_ϕ at $\rho = 0.5$ mm; (*b*) B_ϕ at $\rho = 0.8$ mm; (*c*) the total magnetic flux per unit length inside the conductor; (*d*) the total flux for $\rho < 0.5$ mm; (*e*) the total magnetic flux outside the conductor.

Ans. 1592 A/m; 3.2 mT; 2 μWb/m; 0.5 μWb; ∞

7.6 THE SCALAR AND VECTOR MAGNETIC POTENTIALS

The solution of electrostatic field problems is greatly simplified by the use of the scalar electrostatic potential V. Although this potential possesses a very real physical significance for us, it is mathematically no more than a stepping-stone which allows us to solve a problem by several smaller steps. Given a charge configuration, we may first find the potential and then from it the electric field intensity.

We should question whether or not such assistance is available in magnetic fields. Can we define a potential function which may be found from the current distribution and from which the magnetic fields may be easily determined? Can a scalar magnetic potential be defined, similar to the scalar electrostatic potential? We will show in the next few pages that the answer to the first question is yes, but the second must be answered "sometimes." Let us attack the second question first by assuming the existence of a scalar magnetic potential, which we designate V_m, whose negative gradient gives the magnetic field intensity

$$\mathbf{H} = -\nabla V_m$$

The selection of the negative gradient provides a closer analogy to the electric potential and to problems which we have already solved.

This definition must not conflict with our previous results for the magnetic field, and therefore

$$\nabla \times \mathbf{H} = \mathbf{J} = \nabla \times (-\nabla V_m)$$

However, the curl of the gradient of any scalar is identically zero, a vector identity the proof of which is left for a leisure moment. Therefore, we see that if **H** is to be defined as the gradient of a scalar magnetic potential, then current density must be zero throughout the region in which the scalar magnetic potential is so defined. We then have

$$\boxed{\mathbf{H} = -\nabla V_m \quad (\mathbf{J} = 0)} \tag{43}$$

Because many magnetic problems involve geometries in which the current-carrying conductors occupy a relatively small fraction of the total region of interest, it is evident that a scalar magnetic potential can be useful. The scalar magnetic potential is also applicable in the case of permanent magnets. The dimensions of V_m are obviously amperes.

This scalar potential also satisfies Laplace's equation. In free space,

$$\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot \mathbf{H} = 0$$

and hence

$$\mu_0 \nabla \cdot (-\nabla V_m) = 0$$

or

$$\nabla^2 V_m = 0 \quad (\mathbf{J} = 0) \tag{44}$$

We will see later that V_m continues to satisfy Laplace's equation in homogeneous magnetic materials; it is not defined in any region in which current density is present.

Although we shall consider the scalar magnetic potential to a much greater extent in Chapter 8, when we introduce magnetic materials and discuss the magnetic circuit, one difference between V and V_m should be pointed out now: V_m is not a single-valued function of position. The electric potential V is single-valued; once a zero reference is assigned, there is only one value of V associated with each point in space. Such is not the case with V_m. Consider the cross section of the coaxial line shown in Figure 7.18. In the region $a < \rho < b$, $\mathbf{J} = 0$, and we may establish a scalar magnetic potential. The value of $\mathbf{H}$ is

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi$$

where I is the total current flowing in the $\mathbf{a}_z$ direction in the inner conductor. We find V_m by integrating the appropriate component of the gradient. Applying (43),

$$\frac{I}{2\pi\rho} = -\nabla V_m \Big|_\phi = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}$$

or

$$\frac{\partial V_m}{\partial \phi} = -\frac{I}{2\pi}$$

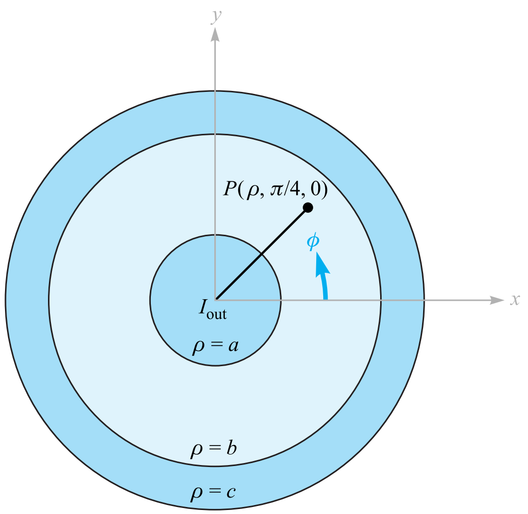

Figure 7.18 The scalar magnetic potential V_m is a multivalued function of ϕ in the region $a < \rho < b$. The electrostatic potential is always single valued.

Thus,

$$V_m = -\frac{I}{2\pi}\phi$$

where the constant of integration has been set equal to zero. What value of potential do we associate with point P, where $\phi = \pi/4$? If we let V_m be zero at $\phi = 0$ and proceed counterclockwise around the circle, the magnetic potential goes negative linearly. When we have made one circuit, the potential is $-I$, but that was the point at which we said the potential was zero a moment ago. At P, then, $\phi = \pi/4, 9\pi/4, 17\pi/4, \ldots,$ or $-7\pi/4, -15\pi/4, -23\pi/4, \ldots,$ or

$$V_{mP} = \frac{I}{2\pi}\left(2n - \tfrac{1}{4}\right)\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$$

or

$$V_{mP} = I\left(n - \tfrac{1}{8}\right) \quad (n = 0, \pm 1, \pm 2, \ldots)$$

The reason for this multivaluedness may be shown by a comparison with the electrostatic case. There, we know that

$$\nabla \times \mathbf{E} = 0$$

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

and therefore the line integral

$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{L}$$

is independent of the path. In the magnetostatic case, however,

$$\nabla \times \mathbf{H} = 0 \quad (\text{wherever } \mathbf{J} = 0)$$

but

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

even if $\mathbf{J}$ is zero along the path of integration. Every time we make another complete lap around the current, the result of the integration increases by I. If no current I is enclosed by the path, then a single-valued potential function may be defined. In general, however,

$$V_{m,ab} = -\int_b^a \mathbf{H} \cdot d\mathbf{L} \quad (\text{specified path}) \tag{45}$$

where a specific path or type of path must be selected. We should remember that the electrostatic potential V is a conservative field; the magnetic scalar potential V_m is not a conservative field. In our coaxial problem, let us erect a barrier[8] at $\phi = \pi$; we

[8] This corresponds to the more precise mathematical term "branch cut."

agree not to select a path that crosses this plane. Therefore, we cannot encircle I, and a single-valued potential is possible. The result is seen to be

$$V_m = -\frac{I}{2\pi}\phi \quad (-\pi < \phi < \pi)$$

and

$$V_{mP} = -\frac{I}{8} \quad \left(\phi = \frac{\pi}{4}\right)$$

The scalar magnetic potential is evidently the quantity whose equipotential surfaces will form curvilinear squares with the streamlines of **H** in Figure 7.4. This is one more facet of the analogy between electric and magnetic fields about which we will have more to say in the next chapter.

Let us temporarily leave the scalar magnetic potential now and investigate a vector magnetic potential. This vector field is one which is extremely useful in studying radiation from antennas (as we will find in Chapter 14) as well as radiation leakage from transmission lines, waveguides, and microwave ovens. The vector magnetic potential may be used in regions where the current density is zero or nonzero, and we shall also be able to extend it to the time-varying case later.

Our choice of a vector magnetic potential is indicated by noting that

$$\nabla \cdot \mathbf{B} = 0$$

Next, a vector identity that we proved in Section 7.4 shows that the divergence of the curl of any vector field is zero. Therefore, we select

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{46}$$

where A signifies a *vector magnetic potential,* and we automatically satisfy the condition that the magnetic flux density shall have zero divergence. The **H** field is

$$\mathbf{H} = \frac{1}{\mu_0}\nabla \times \mathbf{A}$$

and

$$\nabla \times \mathbf{H} = \mathbf{J} = \frac{1}{\mu_0}\nabla \times \nabla \times \mathbf{A}$$

The curl of the curl of a vector field is not zero and is given by a fairly complicated expression,[9] which we need not know now in general form. In specific cases for which the form of **A** is known, the curl operation may be applied twice to determine the current density.

[9] $\nabla \times \nabla \times \mathrm{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathrm{A}$. In rectangular coordinates, it may be shown that $\nabla^2\mathrm{A} \equiv \nabla^2\mathrm{A}_x\mathbf{a}_x + \nabla^2\mathrm{A}_y\mathbf{a}_y + \nabla^2\mathrm{A}_z\mathbf{a}_z$. In other coordinate systems, $\nabla^2\mathrm{A}$ may be found by evaluating the second-order partial derivatives in $\nabla^2\mathrm{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathrm{A}$.

Equation (46) serves as a useful definition of the *vector magnetic potential* **A**. Because the curl operation implies differentiation with respect to a length, the units of **A** are webers per meter.

As yet we have seen only that the definition for **A** does not conflict with any previous results. It still remains to show that this particular definition can help us to determine magnetic fields more easily. We certainly cannot identify **A** with any easily measured quantity or history-making experiment.

We will show in Section 7.7 that, given the Biot-Savart law, the definition of **B**, and the definition of **A**, **A** may be determined from the differential current elements by

$$\mathbf{A} = \oint \frac{\mu_0 I \, d\mathbf{L}}{4\pi R} \tag{47}$$

The significance of the terms in (47) is the same as in the Biot-Savart law; a direct current I flows along a filamentary conductor of which any differential length $d\mathbf{L}$ is distant R from the point at which **A** is to be found. Because we have defined **A** only through specification of its curl, it is possible to add the gradient of any scalar field to (47) without changing **B** or **H**, for the curl of the gradient is identically zero. In steady magnetic fields, it is customary to set this possible added term equal to zero.

The fact that **A** is a vector magnetic *potential* is more apparent when (47) is compared with the similar expression for the electrostatic potential,

$$V = \int \frac{\rho_L dL}{4\pi\epsilon_0 R}$$

Each expression is the integral along a line source, in one case line charge and in the other case line current; each integrand is inversely proportional to the distance from the source to the point of interest; and each involves a characteristic of the medium (here free space), the permeability or the permittivity.

Equation (47) may be written in differential form,

$$d\mathbf{A} = \frac{\mu_0 I \, d\mathbf{L}}{4\pi R} \tag{48}$$

if we again agree not to attribute any physical significance to any magnetic fields we obtain from (48) until the *entire closed path in which the current flows is considered*.

With this reservation, let us go right ahead and consider the vector magnetic potential field about a differential filament. We locate the filament at the origin in free space, as shown in Figure 7.19, and allow it to extend in the positive z direction so that $d\mathbf{L} = dz\,\mathbf{a}_z$. We use cylindrical coordinates to find $d\mathbf{A}$ at the point (ρ, ϕ, z):

$$d\mathbf{A} = \frac{\mu_0 I \, dz\,\mathbf{a}_z}{4\pi\sqrt{\rho^2 + z^2}}$$

or

$$d\mathbf{A}_z = \frac{\mu_0 I \, dz}{4\pi\sqrt{\rho^2 + z^2}} \quad dA_\phi = 0 \quad dA_\rho = 0 \tag{49}$$

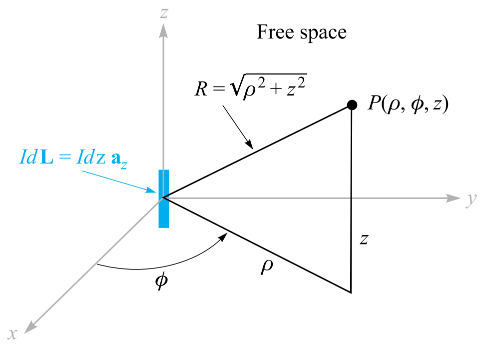

Figure 7.19 The differential current element $I\, dz\mathbf{a}_z$ at the origin establishes the differential vector magnetic potential field, $d\mathbf{A} = \dfrac{\mu_0 I\, dz\mathbf{a}_z}{4\pi\sqrt{\rho^2 + z^2}}$ at $P(\rho, \phi, z)$.

We note that the direction of $d\mathbf{A}$ is the same as that of $I\, d\mathbf{L}$. Each small section of a current-carrying conductor produces a contribution to the total vector magnetic potential which is in the same direction as the current flow in the conductor. The magnitude of the vector magnetic potential varies inversely with the distance to the current element, being strongest in the neighborhood of the current and gradually falling off to zero at distant points. Skilling[10] describes the vector magnetic potential field as "like the current distribution but fuzzy around the edges, or like a picture of the current out of focus."

In order to find the magnetic field intensity, we must take the curl of (49) in cylindrical coordinates, leading to

$$d\mathbf{H} = \frac{1}{\mu_0}\nabla \times d\mathbf{A} = \frac{1}{\mu_0}\left(-\frac{\partial dA_z}{\partial \rho}\right)\mathbf{a}_\phi$$

or

$$d\mathbf{H} = \frac{I\, dz}{4\pi}\frac{\rho}{(\rho^2 + z^2)^{3/2}}\mathbf{a}_\phi$$

which is easily shown to be the same as the value given by the Biot-Savart law.

Expressions for the vector magnetic potential $\mathbf{A}$ can also be obtained for a current source which is distributed. For a current sheet $\mathbf{K}$, the differential current element becomes

$$I\, d\mathbf{L} = \mathbf{K}\, dS$$

In the case of current flow throughout a volume with a density $\mathbf{J}$, we have

$$I\, d\mathbf{L} = \mathbf{J}\, dv$$

[10] See the References at the end of the chapter.

In each of these two expressions the vector character is given to the current. For the filamentary element it is customary, although not necessary, to use $I\,d\mathbf{L}$ instead of $\mathbf{I}\,dL$. Since the magnitude of the filamentary element is constant, we have chosen the form which allows us to remove one quantity from the integral. The alternative expressions for **A** are then

$$\mathbf{A} = \int_S \frac{\mu_0 \mathbf{K}\,dS}{4\pi R} \tag{50}$$

and

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu_0 \mathbf{J}\,dv}{4\pi R} \tag{51}$$

Equations (47), (50), and (51) express the vector magnetic potential as an integration over all of its sources. From a comparison of the form of these integrals with those which yield the electrostatic potential, it is evident that once again the zero reference for **A** is at infinity, for no finite current element can produce any contribution as $R \to \infty$. We should remember that we very seldom used the similar expressions for V; too often our theoretical problems included charge distributions that extended to infinity, and the result would be an infinite potential everywhere. Actually, we calculated very few potential fields until the differential form of the potential equation was obtained, $\nabla^2 V = -\rho_v/\epsilon$, or better yet, $\nabla^2 V = 0$. We were then at liberty to select our own zero reference.

The analogous expressions for **A** will be derived in the next section, and an example of the calculation of a vector magnetic potential field will be completed.

D7.8. A current sheet, $\mathbf{K} = 2.4\mathbf{a}_z$ A/m, is present at the surface $\rho = 1.2$ in free space. (*a*) Find **H** for $\rho > 1.2$. Find V_m at $P(\rho = 1.5, \phi = 0.6\pi, z = 1)$ if: (*b*) $V_m = 0$ at $\phi = 0$ and there is a barrier at $\phi = \pi$; (*c*) $V_m = 0$ at $\phi = 0$ and there is a barrier at $\phi = \pi/2$; (*d*) $V_m = 0$ at $\phi = \pi$ and there is a barrier at $\phi = 0$; (*e*) $V_m = 5$ V at $\phi = \pi$ and there is a barrier at $\phi = 0.8\pi$.

Ans. $\frac{2.88}{\rho}\mathbf{a}_\phi$; −5.43 V; 12.7 V; 3.62 V; −9.48 V

D7.9. The value of **A** within a solid nonmagnetic conductor of radius a carrying a total current I in the $\mathbf{a}_z$ direction may be found easily. Using the known value of **H** or **B** for $\rho < a$, then (46) may be solved for **A**. Select $A = (\mu_0 I \ln 5)/2\pi$ at $\rho = a$ (to correspond with an example in the next section) and find **A** at $\rho =$: (*a*) 0; (*b*) $0.25a$; (*c*) $0.75a$; (*d*) a.

Ans. $0.422I\mathbf{a}_z$ μWb/m; $0.416I\mathbf{a}_z$ μWb/m; $0.366I\mathbf{a}_z$ μWb/m; $0.322I\mathbf{a}_z$ μWb/m

7.7 DERIVATION OF THE STEADY-MAGNETIC-FIELD LAWS

We will now supply the promised proofs of the several relationships between the magnetic field quantities. All these relationships may be obtained from the definitions of **H**,

$$\mathbf{H} = \oint \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} \tag{3}$$

of **B** (in free space),

$$\mathbf{B} = \mu_0 \mathbf{H} \tag{32}$$

and of **A**,

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{46}$$

Let us first assume that we may express **A** by the last equation of Section 7.6,

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu_0 \mathbf{J}\,dv}{4\pi R} \tag{51}$$

and then demonstrate the correctness of (51) by showing that (3) follows. First, we should add subscripts to indicate the point at which the current element is located (x_1, y_1, z_1) and the point at which **A** is given (x_2, y_2, z_2). The differential volume element dv is then written dv_1 and in rectangular coordinates would be $dx_1\,dy_1\,dz_1$. The variables of integration are x_1, y_1, and z_1. Using these subscripts, then,

$$\mathbf{A}_2 = \int_{\text{vol}} \frac{\mu_0 \mathbf{J}_1 dv_1}{4\pi R_{12}} \tag{52}$$

From (32) and (46) we have

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{\nabla \times \mathbf{A}}{\mu_0} \tag{53}$$

To show that (3) follows from (52), it is necessary to substitute (52) into (53). This step involves taking the curl of $\mathbf{A}_2$, a quantity expressed in terms of the variables x_2, y_2, and z_2, and the curl therefore involves partial derivatives with respect to x_2, y_2, and z_2. We do this, placing a subscript on the del operator to remind us of the variables involved in the partial differentiation process,

$$\mathbf{H}_2 = \frac{\nabla_2 \times \mathbf{A}_2}{\mu_0} = \frac{1}{\mu_0} \nabla_2 \times \int_{\text{vol}} \frac{\mu_0 \mathbf{J}_1 dv_1}{4\pi R_{12}}$$

The order of partial differentiation and integration is immaterial, and $\mu_0/4\pi$ is constant, allowing us to write

$$\mathbf{H}_2 = \frac{1}{4\pi} \int_{\text{vol}} \nabla_2 \times \frac{\mathbf{J}_1 dv_1}{R_{12}}$$

The curl operation within the integrand represents partial differentiation with respect to x_2, y_2, and z_2. The differential volume element dv_1 is a scalar and a function only of x_1, y_1, and z_1. Consequently, it may be factored out of the curl operation as any other constant, leaving

$$\mathbf{H}_2 = \frac{1}{4\pi}\int_{\text{vol}}\left(\nabla_2 \times \frac{\mathbf{J}_1}{R_{12}}\right)dv_1 \tag{54}$$

The curl of the product of a scalar and a vector is given by an identity which may be checked by expansion in rectangular coordinates or obtained from Appendix A.3,

$$\nabla \times (S\mathbf{V}) \equiv (\nabla S) \times \mathbf{V} + S(\nabla \times \mathbf{V}) \tag{55}$$

This identity is used to expand the integrand of (54),

$$\mathbf{H}_2 = \frac{1}{4\pi}\int_{\text{vol}}\left[\left(\nabla_2 \frac{1}{R_{12}}\right) \times \mathbf{J}_1 + \frac{1}{R_{12}}(\nabla_2 \times \mathbf{J}_1)\right]dv_1 \tag{56}$$

The second term of this integrand is zero because $\nabla_2 \times \mathbf{J}_1$ indicates partial derivatives of a function of x_1, y_1, and z_1, taken with respect to the variables x_2, y_2, and z_2; the first set of variables is not a function of the second set, and all partial derivatives are zero.

The first term of the integrand may be determined by expressing R_{12} in terms of the coordinate values,

$$R_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

and taking the gradient of its reciprocal. Problem 7.42 shows that the result is

$$\nabla_2 \frac{1}{R_{12}} = -\frac{\mathbf{R}_{12}}{R_{12}^3} = -\frac{\mathbf{a}_{R12}}{R_{12}^2}$$

Substituting this result into (56), we have

$$\mathbf{H}_2 = -\frac{1}{4\pi}\int_{\text{vol}}\frac{\mathbf{a}_{R12} \times \mathbf{J}_1}{R_{12}^2}dv_1$$

or

$$\mathbf{H}_2 = \int_{\text{vol}}\frac{\mathbf{J}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}dv_1$$

which is the equivalent of (3) in terms of current density. Replacing $\mathbf{J}_1\, dv_1$ by $I_1\, d\mathbf{L}_1$, we may rewrite the volume integral as a closed line integral,

$$\mathbf{H}_2 = \oint \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}$$

Equation (51) is therefore correct and agrees with the three definitions (3), (32), and (46).

Next we will prove Ampère's circuital law in point form,

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{28}$$

Combining (28), (32), and (46), we obtain

$$\nabla \times \mathbf{H} = \nabla \times \frac{\mathbf{B}}{\mu_0} = \frac{1}{\mu_0}\nabla \times \nabla \times \mathbf{A} \tag{57}$$

We now need the expansion in rectangular coordinates for $\nabla \times \nabla \times \mathbf{A}$. Performing the indicated partial differentiations and collecting the resulting terms, we may write the result as

$$\nabla \times \nabla \times \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{58}$$

where

$$\nabla^2 \mathbf{A} \equiv \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z \tag{59}$$

Equation (59) is the definition (in rectangular coordinates) of the *Laplacian of a vector*.

Substituting (58) into (57), we have

$$\nabla \times \mathbf{H} = \frac{1}{\mu_0}[\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}] \tag{60}$$

and now require expressions for the divergence and the Laplacian of $\mathbf{A}$.

We may find the divergence of $\mathbf{A}$ by applying the divergence operation to (52),

$$\nabla_2 \cdot \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_{\text{vol}} \nabla_2 \cdot \frac{\mathbf{J}_1}{R_{12}} dv_1 \tag{61}$$

and using the vector identity (44) of Section 4.8,

$$\nabla \cdot (S\mathbf{V}) \equiv \mathbf{V} \cdot (\nabla S) + S(\nabla \cdot \mathbf{V})$$

Thus,

$$\nabla_2 \cdot \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_{\text{vol}} \left[\mathbf{J}_1 \cdot \left(\nabla_2 \frac{1}{R_{12}} \right) + \frac{1}{R_{12}} (\nabla_2 \cdot \mathbf{J}_1) \right] dv_1 \tag{62}$$

The second part of the integrand is zero because $\mathbf{J}_1$ is not a function of x_2, y_2, and z_2.

We have already used the result that $\nabla_2(1/R_{12}) = -\mathbf{R}_{12}/R_{12}^3$, and it is just as easily shown that

$$\nabla_1 \frac{1}{R_{12}} = \frac{\mathbf{R}_{12}}{R_{12}^3}$$

or that

$$\nabla_1 \frac{1}{R_{12}} = -\nabla_2 \frac{1}{R_{12}}$$

Equation (62) can therefore be written as

$$\nabla_2 \cdot \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_{\text{vol}} \left[-\mathbf{J}_1 \cdot \left(\nabla_1 \frac{1}{R_{12}} \right) \right] dv_1$$

and the vector identity applied again,

$$\nabla_2 \cdot \mathbf{A}_2 = \frac{\mu_0}{4\pi} \int_{\text{vol}} \left[\frac{1}{R_{12}} (\nabla_1 \cdot \mathbf{J}_1) - \nabla_1 \cdot \left(\frac{\mathbf{J}_1}{R_{12}} \right) \right] dv_1 \tag{63}$$

Because we are concerned only with steady magnetic fields, the continuity equation shows that the first term of (63) is zero. Application of the divergence theorem to the second term gives

$$\nabla_2 \cdot \mathbf{A}_2 = -\frac{\mu_0}{4\pi} \oint_{S_1} \frac{\mathbf{J}_1}{R_{12}} \cdot d\mathbf{S}_1$$

where the surface S_1 encloses the volume throughout which we are integrating. This volume must include all the current, for the original integral expression for **A** was an integration such as to include the effect of all the current. Because there is no current outside this volume (otherwise we should have had to increase the volume to include it), we may integrate over a slightly larger volume or a slightly larger enclosing surface without changing **A.** On this larger surface the current density $\mathbf{J}_1$ must be zero, and therefore the closed surface integral is zero, since the integrand is zero. Hence the divergence of **A** is zero.

In order to find the Laplacian of the vector **A**, let us compare the x component of (51) with the similar expression for electrostatic potential,

$$A_x = \int_{\text{vol}} \frac{\mu_0 J_x dv}{4\pi R} \qquad V = \int_{\text{vol}} \frac{\rho_v \, dv}{4\pi\epsilon_0 R}$$

We note that one expression can be obtained from the other by a straightforward change of variable, J_x for ρ_v, μ_0 for $1/\epsilon_0$, and A_x for V. However, we have derived some additional information about the electrostatic potential which we shall not have to repeat now for the x component of the vector magnetic potential. This takes the form of Poisson's equation,

$$\nabla^2 V = -\frac{\rho_v}{\epsilon_0}$$

which becomes, after the change of variables,

$$\nabla^2 A_x = -\mu_0 J_x$$

Similarly, we have

$$\nabla^2 A_y = -\mu_0 J_y$$

and

$$\nabla^2 A_z = -\mu_0 J_z$$

or

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{64}$$

Returning to (60), we can now substitute for the divergence and Laplacian of **A** and obtain the desired answer,

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{28}$$

We have already shown the use of Stokes' theorem in obtaining the integral form of Ampère's circuital law from (28) and need not repeat that labor here.

We thus have succeeded in showing that every result we have essentially pulled from thin air[11] for magnetic fields follows from the basic definitions of **H**, **B**, and **A**. The derivations are not simple, but they should be understandable on a step-by-step basis.

Finally, let us return to (64) and make use of this formidable second-order vector partial differential equation to find the vector magnetic potential in one simple example. We select the field between conductors of a coaxial cable, with radii of a and b as usual, and current I in the $\mathbf{a}_z$ direction in the inner conductor. Between the conductors, $\mathbf{J} = 0$, and therefore

$$\nabla^2 \mathbf{A} = 0$$

We have already been told (and Problem 7.44 gives us the opportunity to check the results for ourselves) that the vector Laplacian may be expanded as the vector sum of the scalar Laplacians of the three components in rectangular coordinates,

$$\nabla^2 \mathbf{A} = \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z$$

but such a relatively simple result is not possible in other coordinate systems. That is, in cylindrical coordinates, for example,

$$\nabla^2 \mathbf{A} \neq \nabla^2 A_\rho \mathbf{a}_\rho + \nabla^2 A_\phi \mathbf{a}_\phi + \nabla^2 A_z \mathbf{a}_z$$

However, it is not difficult to show for cylindrical coordinates that the z component of the vector Laplacian is the scalar Laplacian of the z component of **A**, or

$$\nabla^2 \mathbf{A}\Big|_z = \nabla^2 A_z \tag{65}$$

and because the current is entirely in the z direction in this problem, **A** has only a z component. Therefore,

$$\nabla^2 A_z = 0$$

or

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial A_z}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2} = 0$$

Thinking symmetrical thoughts about (51) shows us that A_z is a function only of ρ, and thus

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dA_z}{d\rho}\right) = 0$$

We have solved this equation before, and the result is

$$A_z = C_1 \ln \rho + C_2$$

If we choose a zero reference at $\rho = b$, then

$$A_z = C_1 \ln \frac{\rho}{b}$$

[11] Free space.

In order to relate C_1 to the sources in our problem, we may take the curl of **A**,

$$\nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial \rho}\mathbf{a}_\phi = -\frac{C_1}{\rho}\mathbf{a}_\phi = \mathbf{B}$$

obtain **H**,

$$\mathbf{H} = -\frac{C_1}{\mu_0 \rho}\mathbf{a}_\phi$$

and evaluate the line integral,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_0^{2\pi} -\frac{C_1}{\mu_0 \rho}\mathbf{a}_\phi \cdot \rho\, d\phi\, \mathbf{a}_\phi = -\frac{2\pi C_1}{\mu_0}$$

Thus

$$C_1 = -\frac{\mu_0 I}{2\pi}$$

or

$$A_z = \frac{\mu_0 I}{2\pi} \ln \frac{b}{\rho} \tag{66}$$

and

$$H_\phi = \frac{I}{2\pi\rho}$$

as before. A plot of A_z versus ρ for $b = 5a$ is shown in Figure 7.20; the decrease of $|\mathbf{A}|$ with distance from the concentrated current source that the inner conductor represents is evident. The results of Problem D7.9 have also been added to Figure 7.20. The extension of the curve into the outer conductor is left as Problem 7.43.

It is also possible to find A_z between conductors by applying a process some of us informally call "uncurling." That is, we know **H** or **B** for the coax, and we may

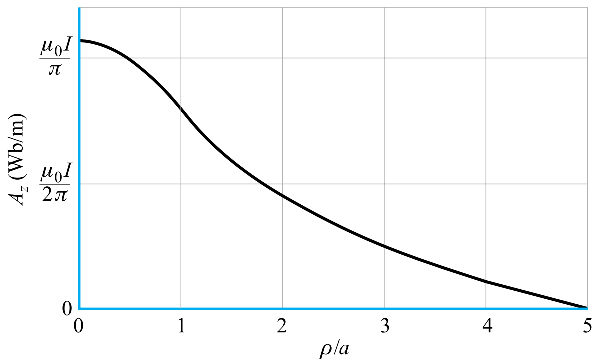

Figure 7.20 The vector magnetic potential is shown within the inner conductor and in the region between conductors for a coaxial cable with $b = 5a$ carrying I in the $\mathbf{a}_z$ direction. $A_z = 0$ is arbitrarily selected at $\rho = b$.

therefore select the ϕ component of $\nabla \times \mathbf{A} = \mathbf{B}$ and integrate to obtain A_z. Try it, you'll like it!

D7.10. Equation (66) is obviously also applicable to the exterior of any conductor of circular cross section carrying a current I in the $\mathbf{a}_z$ direction in free space. The zero reference is arbitrarily set at $\rho = b$. Now consider two conductors, each of 1 cm radius, parallel to the z axis with their axes lying in the $x = 0$ plane. One conductor whose axis is at $(0, 4\,\text{cm}, z)$ carries 12 A in the $\mathbf{a}_z$ direction; the other axis is at $(0, -4\text{ cm}, z)$ and carries 12 A in the $-\mathbf{a}_z$ direction. Each current has its zero reference for $\mathbf{A}$ located 4 cm from its axis. Find the total $\mathbf{A}$ field at: (*a*) $(0, 0, z)$; (*b*) $(0, 8\,\text{cm}, z)$; (*c*) $(4\,\text{cm},\ 4\,\text{cm}, z)$; (*d*) $(2\,\text{cm}, 4\,\text{cm}, z)$.

Ans. 0; 2.64 μWb/m; 1.93 μWb/m; 3.40 μWb/m

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1. Boast, W. B. (See References for Chapter 2.) The scalar magnetic potential is defined on p. 220, and its use in mapping magnetic fields is discussed on p. 444.
2. Jordan, E. C., and K. G. Balmain. *Electromagnetic Waves and Radiating Systems.* 2d ed. Englewood Cliffs, N.J.: Prentice-Hall, 1968. Vector magnetic potential is discussed on pp. 90–96.
3. Paul, C. R., K. W. Whites, and S. Y. Nasar. *Introduction to Electromagnetic Fields*. 3d ed. New York: McGraw-Hill, 1998. The vector magnetic potential is presented on pp. 216–20.
4. Skilling, H. H. (See References for Chapter 3.) The "paddle wheel" is introduced on pp. 23–25.

CHAPTER 7 PROBLEMS

WWW Quizzes

7.1 (*a*) Find $\mathbf{H}$ in rectangular components at $P(2, 3, 4)$ if there is a current filament on the z axis carrying 8 mA in the $\mathbf{a}_z$ direction. (*b*) Repeat if the filament is located at $x = -1$, $y = 2$. (*c*) Find $\mathbf{H}$ if both filaments are present.

7.2 A filamentary conductor is formed into an equilateral triangle with sides of length ℓ carrying current I. Find the magnetic field intensity at the center of the triangle.

7.3 Two semi-infinite filaments on the z axis lie in the regions $-\infty < z < -a$ and $a < z < \infty$. Each carries a current I in the $\mathbf{a}_z$ direction. (*a*) Calculate $\mathbf{H}$ as a function of ρ and ϕ at $z = 0$. (*b*) What value of a will cause the magnitude of $\mathbf{H}$ at $\rho = 1$, $z = 0$, to be one-half the value obtained for an infinite filament?

7.4 Two circular current loops are centered on the z axis at $z = \pm h$. Each loop has radius a and carries current I in the $\mathbf{a}_\phi$ direction. (*a*) Find $\mathbf{H}$ on the z axis over the range $-h < z < h$. Take $I = 1$ A and plot $|\mathbf{H}|$ as a function of z/a if

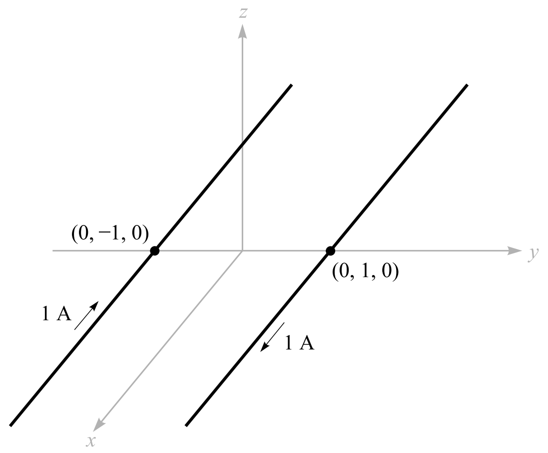

Figure 7.21 See Problem 7.5.

(*b*) $h = a/4$; (*c*) $h = a/2$; (*d*) $h = a$. Which choice for h gives the most uniform field? These are called Helmholtz coils (of a single turn each in this case), and are used in providing uniform fields.

7.5 The parallel filamentary conductors shown in Figure 7.21 lie in free space. Plot $|\mathbf{H}|$ versus y, $-4 < y < 4$, along the line $x = 0$, $z = 2$.

7.6 A disk of radius a lies in the xy plane, with the z axis through its center. Surface charge of uniform density ρ_s lies on the disk, which rotates about the z axis at angular velocity Ω rad/s. Find $\mathbf{H}$ at any point on the z axis.

7.7 A filamentary conductor carrying current I in the $\mathbf{a}_z$ direction extends along the entire negative z axis. At $z = 0$ it connects to a copper sheet that fills the $x > 0$, $y > 0$ quadrant of the xy plane. (*a*) Set up the Biot-Savart law and find $\mathbf{H}$ everywhere on the z axis; (*b*) repeat part (*a*), but with the copper sheet occupying the *entire* xy plane (Hint: express $\mathbf{a}_\phi$ in terms of $\mathbf{a}_x$ and $\mathbf{a}_y$ and angle ϕ in the integral).

7.8 For the finite-length current element on the z axis, as shown in Figure 7.5, use the Biot-Savart law to derive Eq. (9) of Section 7.1.

7.9 A current sheet $\mathbf{K} = 8\mathbf{a}_x$ A/m flows in the region $-2 < y < 2$ in the plane $z = 0$. Calculate H at $P(0, 0, 3)$.

7.10 A hollow spherical conducting shell of radius a has filamentary connections made at the top ($r = a, \theta = 0$) and bottom ($r = a, \theta = \pi$). A direct current I flows down the upper filament, down the spherical surface, and out the lower filament. Find $\mathbf{H}$ in spherical coordinates (*a*) inside and (*b*) outside the sphere.

7.11 An infinite filament on the z axis carries 20π mA in the $\mathbf{a}_z$ direction. Three $\mathbf{a}_z$-directed uniform cylindrical current sheets are also present: 400 mA/m at

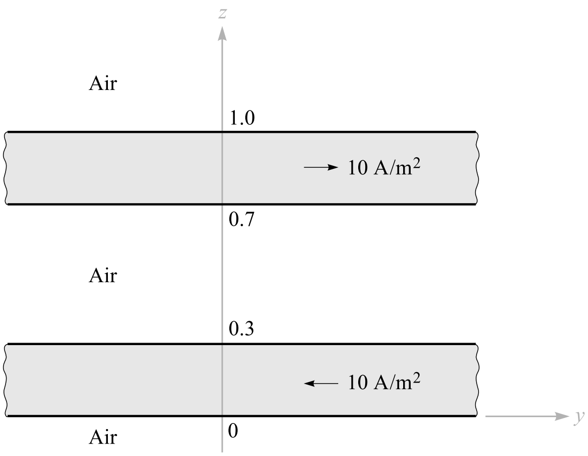

Figure 7.22 See Problem 7.12.

$\rho = 1$ cm, -250 mA/m at $\rho = 2$ cm, and -300 mA/m at $\rho = 3$ cm. Calculate H_ϕ at $\rho = 0.5$, 1.5, 2.5, and 3.5 cm.

7.12 In Figure 7.22, let the regions $0 < z < 0.3$ m and $0.7 < z < 1.0$ m be conducting slabs carrying uniform current densities of 10 A/m^2 in opposite directions as shown. Find **H** at $z =$: (*a*) -0.2; (*b*) 0.2; (*c*) 0.4; (*d*) 0.75; (*e*) 1.2 m.

7.13 A hollow cylindrical shell of radius a is centered on the z axis and carries a uniform surface current density of $K_a \mathbf{a}_\phi$. (*a*) Show that H is not a function of ϕ or z. (*b*) Show that H_ϕ and H_ρ are everywhere zero. (*c*) Show that $H_z = 0$ for $\rho > a$. (*d*) Show that $H_z = K_a$ for $\rho < a$. (*e*) A second shell, $\rho = b$, carries a current $K_b \mathbf{a}_\phi$. Find **H** everywhere.

7.14 A toroid having a cross section of rectangular shape is defined by the following surfaces: the cylinders $\rho = 2$ and $\rho = 3$ cm, and the planes $z = 1$ and $z = 2.5$ cm. The toroid carries a surface current density of $-50\mathbf{a}_z$ A/m on the surface $\rho = 3$ cm. Find **H** at the point $P(\rho, \phi, z)$: (*a*) $P_A(1.5\,\text{cm}, 0, 2\,\text{cm})$; (*b*) $P_B(2.1\,\text{cm}, 0, 2\,\text{cm})$; (*c*) $P_C(2.7\,\text{cm}, \pi/2, 2\,\text{cm})$; (*d*) $P_D(3.5\,\text{cm}, \pi/2, 2\,\text{cm})$.

7.15 Assume that there is a region with cylindrical symmetry in which the conductivity is given by $\sigma = 1.5e^{-150\rho}$ kS/m. An electric field of $30\mathbf{a}_z$ V/m is present. (*a*) Find **J**. (*b*) Find the total current crossing the surface $\rho < \rho_0$, $z = 0$, all ϕ. (*c*) Make use of Ampère's circuital law to find **H**.

7.16 A current filament carrying I in the $-\mathbf{a}_z$ direction lies along the entire positive z axis. At the origin, it connects to a conducting sheet that forms the xy plane. (*a*) Find **K** in the conducting sheet. (*b*) Use Ampere's circuital law to find **H** everywhere for $z > 0$; (*c*) find **H** for $z < 0$.

7.17 A current filament on the z axis carries a current of 7 mA in the $\mathbf{a}_z$ direction, and current sheets of 0.5 $\mathbf{a}_z$ A/m and -0.2 $\mathbf{a}_z$ A/m are located at $\rho = 1$ cm and $\rho = 0.5$ cm, respectively. Calculate $\mathbf{H}$ at: (a) $\rho = 0.5$ cm; (b) $\rho = 1.5$ cm; (c) $\rho = 4$ cm. (d) What current sheet should be located at $\rho = 4$ cm so that $\mathbf{H} = 0$ for all $\rho > 4$ cm?

7.18 A wire of 3 mm radius is made up of an inner material ($0 < \rho < 2$ mm) for which $\sigma = 10^7$ S/m, and an outer material (2 mm $< \rho <$ 3 mm) for which $\sigma = 4 \times 10^7$ S/m. If the wire carries a total current of 100 mA dc, determine $\mathbf{H}$ everywhere as a function of ρ.

7.19 In spherical coordinates, the surface of a solid conducting cone is described by $\theta = \pi/4$ and a conducting plane by $\theta = \pi/2$. Each carries a total current I. The current flows as a surface current radially inward on the plane to the vertex of the cone, and then flows radially outward throughout the cross section of the conical conductor. (a) Express the surface current density as a function of r; (b) express the volume current density inside the cone as a function of r; (c) determine $\mathbf{H}$ as a function of r and θ in the region between the cone and the plane; (d) determine $\mathbf{H}$ as a function of r and θ inside the cone.

7.20 A solid conductor of circular cross section with a radius of 5 mm has a conductivity that varies with radius. The conductor is 20 m long, and there is a potential difference of 0.1 V dc between its two ends. Within the conductor, $\mathbf{H} = 10^5 \rho^2 \mathbf{a}_\phi$ A/m. (a) Find σ as a function of ρ. (b) What is the resistance between the two ends?

7.21 A cylindrical wire of radius a is oriented with the z axis down its center line. The wire carries a nonuniform current down its length of density $\mathbf{J} = b\rho\, \mathbf{a}_z$ A/m^2 where b is a constant. (a) What total current flows in the wire? (b) Find $\mathbf{H}_{in}$ ($0 < \rho < a$), as a function of ρ; (c) find $\mathbf{H}_{out}(\rho > a)$, as a function of ρ; (d) verify your results of parts (b) and (c) by using $\nabla \times \mathbf{H} = \mathbf{J}$.

7.22 A solid cylinder of radius a and length L, where $L \gg a$, contains volume charge of uniform density ρ_0 C/m^3. The cylinder rotates about its axis (the z axis) at angular velocity Ω rad/s. (a) Determine the current density $\mathbf{J}$ as a function of position within the rotating cylinder. (b) Determine $\mathbf{H}$ on-axis by applying the results of Problem 7.6. (c) Determine the magnetic field intensity $\mathbf{H}$ inside and outside. (d) Check your result of part (c) by taking the curl of $\mathbf{H}$.

7.23 Given the field $\mathbf{H} = 20\rho^2 \mathbf{a}_\phi$ A/m: (a) Determine the current density $\mathbf{J}$. (b) Integrate $\mathbf{J}$ over the circular surface $\rho \le 1, 0 < \phi < 2\pi, z = 0$, to determine the total current passing through that surface in the $\mathbf{a}_z$ direction. (c) Find the total current once more, this time by a line integral around the circular path $\rho = 1, 0 < \phi < 2\pi, z = 0$.

7.24 Infinitely long filamentary conductors are located in the $y = 0$ plane at $x = n$ meters where $n = 0, \pm 1, \pm 2, \ldots$ Each carries 1 A in the $\mathbf{a}_z$ direction.

(a) Find $\mathbf{H}$ on the y axis. As a help,

$$\sum_{n=1}^{\infty} \frac{y}{y^2+n^2} = \frac{\pi}{2} - \frac{1}{2y} + \frac{\pi}{e^{2\pi y}-1}$$

(b) Compare your result of part (a) to that obtained if the filaments are replaced by a current sheet in the $y = 0$ plane that carries surface current density $\mathbf{K} = 1\mathbf{a}_z$ A/m.

7.25 When x, y, and z are positive and less than 5, a certain magnetic field intensity may be expressed as $\mathbf{H} = [x^2yz/(y+1)]\mathbf{a}_x + 3x^2z^2\mathbf{a}_y - [xyz^2/(y+1)]\mathbf{a}_z$. Find the total current in the $\mathbf{a}_x$ direction that crosses the strip $x = 2$, $1 \le y \le 4$, $3 \le z \le 4$, by a method utilizing: (a) a surface integral; (b) a closed line integral.

7.26 Consider a sphere of radius $r = 4$ centered at (0, 0, 3). Let S_1 be that portion of the spherical surface that lies above the xy plane. Find $\int_{S_1} (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$ if $\mathbf{H} = 3\rho\,\mathbf{a}_\phi$ in cylindrical coordinates.

7.27 The magnetic field intensity is given in a certain region of space as $\mathbf{H} = [(x+2y)/z^2]\mathbf{a}_y + (2/z)\mathbf{a}_z$ A/m. (a) Find $\nabla \times \mathbf{H}$. (b) Find $\mathbf{J}$. (c) Use $\mathbf{J}$ to find the total current passing through the surface $z = 4$, $1 \le x \le 2$, $3 \le z \le 5$, in the $\mathbf{a}_z$ direction. (d) Show that the same result is obtained using the other side of Stokes' theorem.

7.28 Given $\mathbf{H} = (3r^2/\sin\theta)\mathbf{a}_\theta + 54r\cos\theta\,\mathbf{a}_\phi$ A/m in free space: (a) Find the total current in the $\mathbf{a}_\theta$ direction through the conical surface $\theta = 20^\circ$, $0 \le \phi \le 2\pi$, $0 \le r \le 5$, by whatever side of Stokes' theorem you like the best. (b) Check the result by using the other side of Stokes' theorem.

7.29 A long, straight, nonmagnetic conductor of 0.2 mm radius carries a uniformly distributed current of 2 A dc. (a) Find $\mathbf{J}$ within the conductor. (b) Use Ampère's circuital law to find $\mathbf{H}$ and $\mathbf{B}$ within the conductor. (c) Show that $\nabla \times \mathbf{H} = \mathbf{J}$ within the conductor. (d) Find $\mathbf{H}$ and $\mathbf{B}$ outside the conductor. (e) Show that $\nabla \times \mathbf{H} = \mathbf{J}$ outside the conductor.

7.30 (An inversion of Problem 7.20.) A solid, nonmagnetic conductor of circular cross section has a radius of 2 mm. The conductor is inhomogeneous, with $\sigma = 10^6(1 + 10^6\rho^2)$ S/m. If the conductor is 1 m in length and has a voltage of 1 mV between its ends, find: (a) $\mathbf{H}$ inside; (b) the total magnetic flux inside the conductor.

7.31 The cylindrical shell defined by 1 cm $< \rho <$ 1.4 cm consists of a nonmagnetic conducting material and carries a total current of 50 A in the $\mathbf{a}_z$ direction. Find the total magnetic flux crossing the plane $\phi = 0$, $0 < z < 1$: (a) $0 < \rho <$ 1.2 cm; (b) 1.0 cm $< \rho <$ 1.4 cm; (c) 1.4 cm $< \rho <$ 20 cm.

7.32 The free space region defined by $1 < z < 4$ cm and $2 < \rho < 3$ cm is a toroid of rectangular cross section. Let the surface at $\rho = 3$ cm carry a surface current $\mathbf{K} = 2\mathbf{a}_z$ kA/m. (a) Specify the current densities on the surfaces at

$\rho = 2$ cm, $z = 1$ cm, and $z = 4$ cm. (*b*) Find **H** everywhere. (*c*) Calculate the total flux within the toroid.

7.33 Use an expansion in rectangular coordinates to show that the curl of the gradient of any scalar field G is identically equal to zero.

7.34 A filamentary conductor on the z axis carries a current of 16 A in the $\mathbf{a}_z$ direction, a conducting shell at $\rho = 6$ carries a total current of 12 A in the $-\mathbf{a}_z$ direction, and another shell at $\rho = 10$ carries a total current of 4 A in the $-\mathbf{a}_z$ direction. (*a*) Find **H** for $0 < \rho < 12$. (*b*) Plot H_ϕ versus ρ. (*c*) Find the total flux Φ crossing the surface $1 < \rho < 7$, $0 < z < 1$, at fixed ϕ.

7.35 A current sheet, $\mathbf{K} = 20\,\mathbf{a}_z$ A/m, is located at $\rho = 2$, and a second sheet, $\mathbf{K} = -10\mathbf{a}_z$ A/m, is located at $\rho = 4$. (*a*) Let $V_m = 0$ at $P(\rho = 3, \phi = 0, z = 5)$ and place a barrier at $\phi = \pi$. Find $V_m(\rho, \phi, z)$ for $-\pi < \phi < \pi$. (*b*) Let $\mathbf{A} = 0$ at P and find $\mathbf{A}(\rho, \phi, z)$ for $2 < \rho < 4$.

7.36 Let $\mathbf{A} = (3y - z)\mathbf{a}_x + 2xz\mathbf{a}_y$ Wb/m in a certain region of free space. (*a*) Show that $\nabla \cdot \mathbf{A} = 0$. (*b*) At $P(2, -1, 3)$, find **A**, **B**, **H**, and **J**.

7.37 Let $N = 1000$, $I = 0.8$ A, $\rho_0 = 2$ cm, and $a = 0.8$ cm for the toroid shown in Figure 7.12*b*. Find V_m in the interior of the toroid if $V_m = 0$ at $\rho = 2.5$ cm, $\phi = 0.3\pi$. Keep ϕ within the range $0 < \phi < 2\pi$.

7.38 A square filamentary differential current loop, dL on a side, is centered at the origin in the $z = 0$ plane in free space. The current I flows generally in the $\mathbf{a}_\phi$ direction. (*a*) Assuming that $r >> dL$, and following a method similar to that in Section 4.7, show that

$$d\mathbf{A} = \frac{\mu_0 I (dL)^2 \sin\theta}{4\pi r^2}\,\mathbf{a}_\phi$$

(*b*) Show that

$$d\mathbf{H} = \frac{I(dL)^2}{4\pi r^3}\,(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

The square loop is one form of a *magnetic dipole*.

7.39 Planar current sheets of $\mathbf{K} = 30\mathbf{a}_z$ A/m and $-30\mathbf{a}_z$ A/m are located in free space at $x = 0.2$ and $x = -0.2$, respectively. For the region $-0.2 < x < 0.2$ (*a*) find **H**; (*b*) obtain an expression for V_m if $V_m = 0$ at $P(0.1, 0.2, 0.3)$; (*c*) find **B**; (*d*) obtain an expression for **A** if $\mathbf{A} = 0$ at P.

7.40 Show that the line integral of the vector potential **A** about any closed path is equal to the magnetic flux enclosed by the path, or $\oint \mathbf{A} \cdot d\mathbf{L} = \int \mathbf{B} \cdot d\mathbf{S}$.

7.41 Assume that $\mathbf{A} = 50\rho^2\mathbf{a}_z$ Wb/m in a certain region of free space. (*a*) Find **H** and **B**. (*b*) Find **J**. (*c*) Use **J** to find the total current crossing the surface $0 \le \rho \le 1$, $0 \le \phi < 2\pi$, $z = 0$. (*d*) Use the value of H_ϕ at $\rho = 1$ to calculate $\oint \mathbf{H} \cdot d\mathbf{L}$ for $\rho = 1$, $z = 0$.

7.42 Show that $\nabla_2(1/R_{12}) = -\nabla_1(1/R_{12}) = \mathbf{R}_{21}/R_{12}^3$.

7.43 Compute the vector magnetic potential within the outer conductor for the coaxial line whose vector magnetic potential is shown in Figure 7.20 if the outer radius of the outer conductor is $7a$. Select the proper zero reference and sketch the results on the figure.

7.44 By expanding Eq. (58), Section 7.7 in rectangular coordinates, show that (59) is correct.