Tikrit university

Collage of Engineering Shirqat

Department of Electrical Engineering

Fourth Class- Semester-2

Control Engineering

Chapter 9&10

Lecture 5

Frequency response & Bode plots

Prepared by

Asst Lecturer. Ahmed Saad Names

9.5 Asymptotic Approximations: Bode Plots

The log-magnitude and phase frequency response curves as functions of log ω are called Bode plots or Bode diagrams. Sketching Bode plots can be simplified because they can be approximated as a sequence of straight lines. Straight-line approximations simplify the evaluation of the magnitude and phase frequency response.

Consider the following transfer function:

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_k)}{s^m(s+p_1)(s+p_2)\cdots(s+p_n)} \qquad 9.27$$

The magnitude frequency response is the product of the magnitude frequency responses of each term, or

$$|G(j\omega)| = \left.\frac{K|(s+z_1)||(s+z_2)|\cdots|(s+z_k)|}{|s^m||(s+p_1)||(s+p_2)|\cdots|(s+p_n)|}\right|_{s\to j\omega} \qquad 9.28$$

Thus, if we know the magnitude response of each pole and zero term, we can find the total magnitude response. The process can be simplified by working with the logarithm of the magnitude, since the zero terms' magnitude responses would be added and the pole terms' magnitude responses subtracted, rather than, respectively, multiplied or divided, to yield the logarithm of the total magnitude response. Converting the magnitude response into dB, we obtain

$$\begin{aligned}20\log|G(j\omega)| &= 20\log K + 20\log|(s+z_1)| + 20\log|(s+z_2)| \\ &\quad + \cdots - 20\log|s^m| - 20\log|(s+p_1)| - \cdots|_{s\to j\omega}\end{aligned} \qquad 9.29$$

Thus, if we knew the response of each term, the algebraic sum would yield the total response in dB. Further, if we could make an approximation of each term that would consist only of straight lines, graphical addition of terms would be greatly simplified.

Before proceeding, let us look at the phase response. From Eq. (9.27), the phase frequency response is the *sum* of the phase frequency response curves of the zero terms minus the *sum* of the phase frequency response curves of the pole terms. Again, since the phase response is the sum of individual terms, straight-line approximations to these individual responses simplify graphical addition.

Let us now show how to approximate the frequency response of simple pole and zero terms by straight-line approximations. Later we show how to combine these responses to sketch the frequency response of more complicated functions. In subsequent sections, after a discussion of the Nyquist stability criterion, we learn how to use the Bode plots for the analysis and design of stability and transient response.

9.5.1 Bode Plots for G(s) = (s + a)

Consider a function, G(s) = (s + a), for which we want to sketch separate logarithmic magnitude and phase response plots. Letting s = jω, we have

$$G(j\omega) = (j\omega + a) = a\left(j\frac{\omega}{a} + 1\right) \tag{9.30}$$

At low frequencies when ω approaches zero, $G(j\omega) \approx a$

The magnitude response in dB is 20 logM = 20 log a

where $M = |G(j\omega)|$ and is a constant. Equation 20 log a is shown plotted in Figure 9.6(a) from $\omega = 0.01a$ to a.

At high frequencies where ω ≫ a, Eq. (9.30) becomes

$$G(j\omega) \approx a\left(\frac{j\omega}{a}\right) = a\left(\frac{\omega}{a}\right)\angle 90^\circ = \omega\angle 90^\circ \tag{9.31}$$

The magnitude response in dB is

$$20\log M = 20\log a + 20\log\frac{\omega}{a} = 20\log\omega \tag{9.32}$$

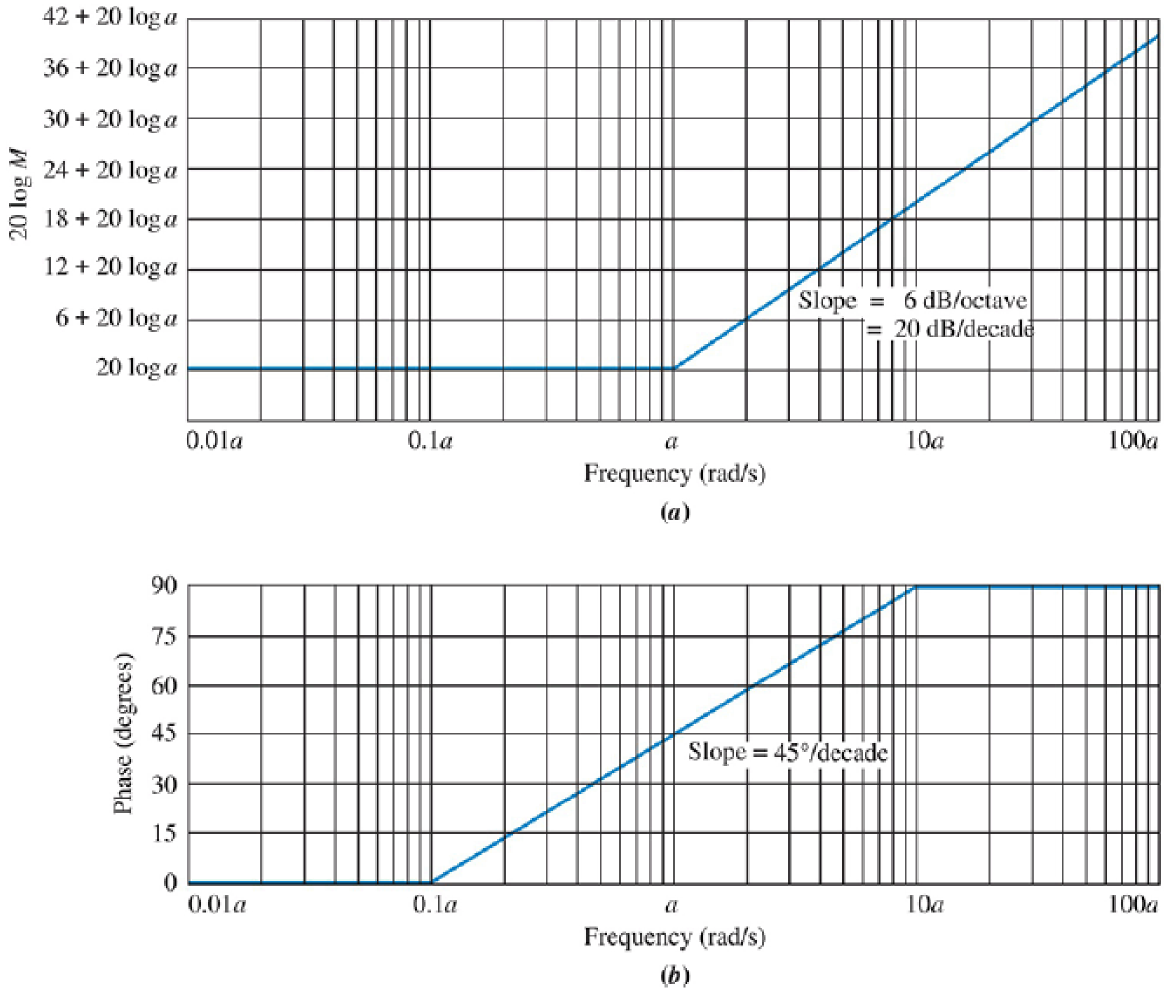


FIGURE 9.6 Bode plots of (s + a): a. magnitude plot; b. phase plot

where $a < \omega < \infty$. Notice from the middle term that the high-frequency approximation is equal to the low-frequency approximation when $\omega = a$, and increases for $\omega > a$. If we plot dB, 20 log M, against log ω, Eq. (9.32) becomes a straight line: y = 20x where $y = 20 \log M$, and $x = \log \omega$. The line has a slope of 20 when plotted as dB vs. log ω.

Since each doubling of frequency causes $20 \log \omega$ to increase by 6 dB, the line rises at an equivalent slope of 6 dB/octave, where an ***octave*** is a doubling of frequency. This rise begins at $\omega = a$, where the low-frequency approximation equals the high-frequency approximation.

We call the straight-line approximations *asymptotes*. The low-frequency approximation is called the *low-frequency asymptote*, and the high-frequency approximation is called the *high-frequency asymptote*. The frequency, *a*, is called the ***break frequency*** because it is the break between the low- and the high-frequency asymptotes.

Many times it is convenient to draw the line over a decade rather than an octave, where a ***decade*** is 10 times the initial frequency. Over one decade, 20 log ω increases by 20 dB. Thus, a slope of 6 dB/octave is equivalent to a slope of 20 dB/decade. The plot is shown in Figure 9.6(*a*) from $\omega = 0.01a$ to $100a$. Let us now turn to the phase response, which can be drawn as follows. At the break frequency, *a*, Eq. (9.30) shows the phase to be 45°. At low frequencies, Eq. G(jw) ≈ a. shows that the phase is 0°. At high frequencies, Eq. (9.31) shows that the phase is 90°. To draw the curve, start one decade (1/10) below the break frequency, 0.1*a*, with 0° phase, and draw a line of slope +45°/decade passing through 45° at the break frequency and continuing to 90° one decade above the break frequency, 10*a*. The resulting phase diagram is shown in Figure 9.6(*b*).

It is often convenient to *normalize* the magnitude and *scale* the frequency so that the log-magnitude plot will be 0 dB at a break frequency of unity. Normalizing and scaling helps in the following applications:

1. *When comparing different first- or second-order frequency response plots, each plot will have the same low-frequency asymptote after normalization and the same break frequency after scaling.*

2. When sketching the frequency response of a function such as Eq. (9.27), each factor in the numerator and denominator will have the same low frequency asymptote after normalization. This common low-frequency asymptote makes it easier to add components to obtain the Bode plot.

To normalize $(s + a)$, we factor out the quantity a and form $a\,[(s/a) + 1]$. The frequency is scaled by defining a new frequency variable, $s1 = s/a$. Then the magnitude is divided by the quantity a to yield 0 dB at the break frequency. Hence, the normalized and scaled function is $(s^1 + 1)$. To obtain the original

frequency response, the magnitude and frequency are multiplied by the quantity a. We now use the concepts of normalization and scaling to compare the asymptotic approximation to the actual magnitude and phase plot for $(s + a)$. Table 9.1 shows the comparison for the normalized and scaled frequency response of $(s + a)$. Notice that the actual magnitude curve is never greater than 3.01 dB from the asymptotes. This maximum difference occurs at the break frequency. The maximum difference for the phase curve is 5.71°, which occurs at the decades above and below the break frequency. For convenience, the data in Table 9.1 is plotted in Figures 9.7 and 9.8.

Table 9.1

Asymptotic and actual normalized and scaled frequency response data for $(s + a)$

Frequency $\frac{}{a}$	$20 \log \frac{M}{a}$ (dB)		Phase (degrees)	
(rad/s)	Asymptotic	Actual	Asymptotic	Actual
0.01	0	0.00	0.00	0.57
0.02	0	0.00	0.00	1.15
0.04	0	0.01	0.00	2.29
0.06	0	0.02	0.00	3.43
0.08	0	0.03	0.00	4.57
0.1	0	0.04	0.00	5.71
0.2	0	0.17	13.55	11.31
0.4	0	0.64	27.09	21.80
0.6	0	1.34	35.02	30.96
0.8	0	2.15	40.64	38.66
1	0	3.01	45.00	45.00
2	6	6.99	58.55	63.43
4	12	12.30	72.09	75.96
6	15.56	15.68	80.02	80.54
8	18	18.13	85.64	82.87
10	20	20.04	90.00	84.29
20	26.02	26.03	90.00	87.14
40	32.04	32.04	90.00	88.57
60	35.56	35.56	90.00	89.05
80	38.06	38.06	90.00	89.28
100	40	40.00	90.00	89.43

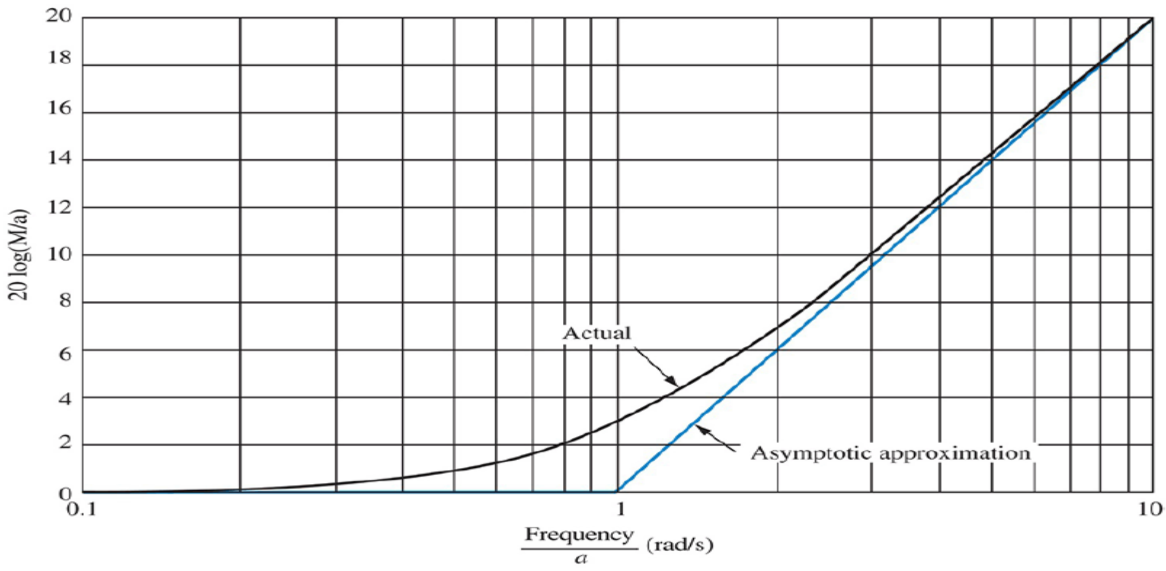


FIGURE 10.7 Asymptotic and actual normalized and scaled magnitude response of (*s* + *a*)

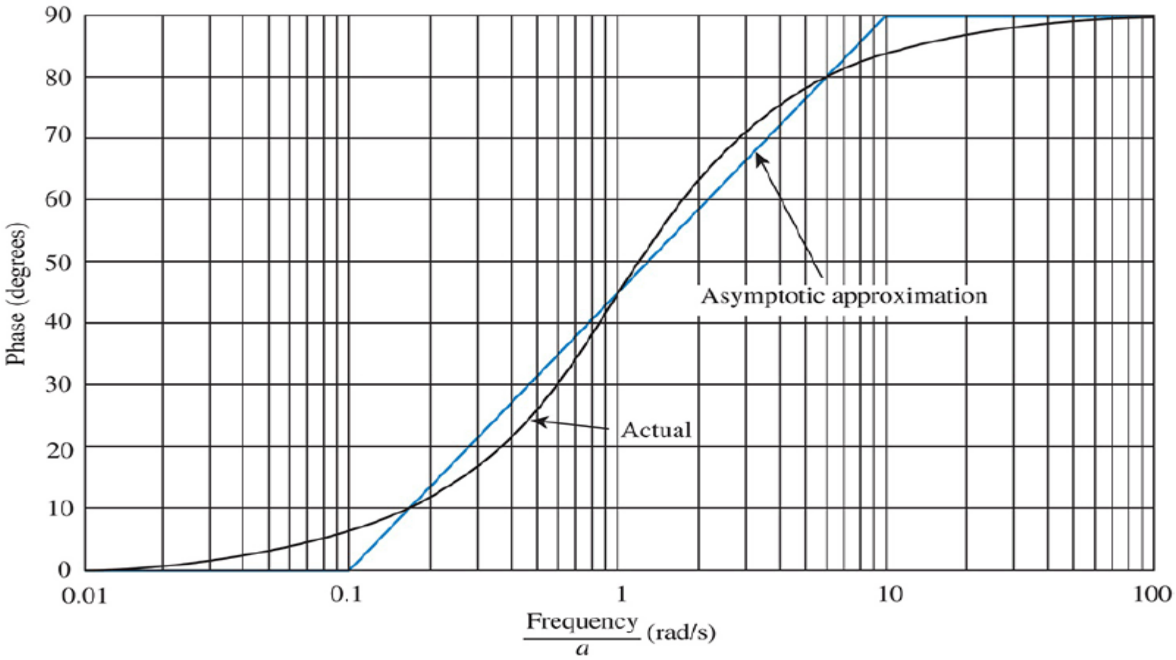


FIGURE 19.8 Asymptotic and actual normalized and scaled phase response of (*s* + *a*)

We now find the Bode plots for other common transfer functions.

9.5.2 Bode Plots for $G(s) = 1/(s + a)$

Let us find the Bode plots for the transfer function

$$G(s) = \frac{1}{(s+a)} = \frac{1}{a\left(\frac{s}{a}+1\right)} \qquad 9.33$$

This function has a low-frequency asymptote of 20 log (1/*a*), which is found by

1. letting the frequency, *s*, approach zero. The Bode plot is constant until the break frequency, *a* rad/s, is reached.

2. The plot is then approximated by the high-frequency asymptote found by letting *s* approach ∞. Thus, at high frequencies,

$$G(j\omega) = \frac{1}{a\left(\frac{s}{a}\right)}\Bigg|_{s \to j\omega} = \frac{1}{a\left(\frac{j\omega}{a}\right)} = \frac{\frac{1}{a}}{\frac{\omega}{a}}\angle -90^\circ = \frac{1}{\omega}\angle -90^\circ \qquad 9.34$$

or, in dB,

$$20 \log M = 20 \log \frac{1}{a} - 20 \log \frac{\omega}{a} = -20 \log \omega \qquad 9.35$$

Notice from the middle term that the high-frequency approximation equals the low-frequency approximation when $\omega = a$, and decreases for $\omega > a$. This result is similar to Eq. (9.31), except the slope is negative rather than positive. The Bode log-magnitude diagram will decrease at a rate of 20dB/decade rather than increase at a rate of 20 dB/decade after the break frequency.

The phase plot is the negative of the previous example, since the function is the inverse. The phase begins at 0° and reaches −90° at high frequencies, going through −45° at the break frequency. Both the Bode normalized and scaled log-magnitude and phase plot are shown in Figure 9.9(*d*).

9.5.3 Bode Plots for *G*(*s*) = *s*

Our next function, $G(s) = s$, has only a high-frequency asymptote. Letting $s = j\omega$, the magnitude is 20 log ω, which is the same as Eq. (9.31). Hence, the Bode magnitude plot is a straight line drawn with a +20-dB/decade slope passing through 0 dB when $\omega = 1$. The phase plot, which is a constant +90°, is shown with the magnitude plot in Figure 9.9(*a*).

9.5.4 Bode Plots for *G*(*s*) = 1/*s*

The frequency response of the inverse of the preceding function, $G(s) = 1/s$, is shown in Figure 9.9(*b*) and is a straight line with a −20 dB/decade slope passing through zero dB at $\omega = 1$. The Bode phase plot is equal to a constant −90°.

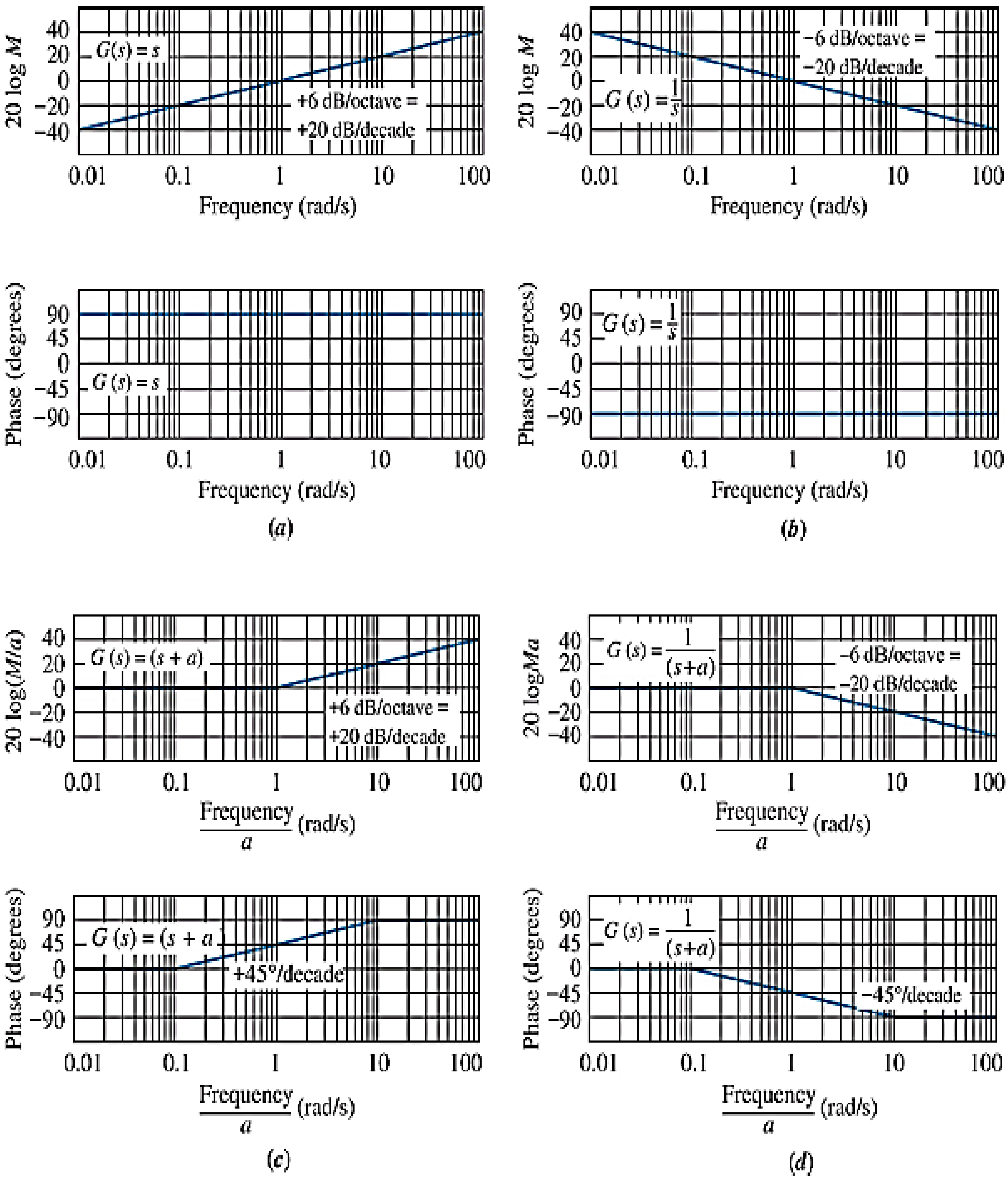


FIGURE 9.9 Normalized and scaled Bode plots for

a. $G(s) = s$;

b. $G(s) = 1/s$;

c. $G(s) = (s + a)$;

d. $G(s) = 1/(s + a)$

Example 9.3 Bode Plots for Ratio of First-Order

Factors

PROBLEM:

Draw the Bode plots for the system shown in Figure 9.10, where $G(s) = K(s + 3)/[s(s + 1)(s + 2)]$.

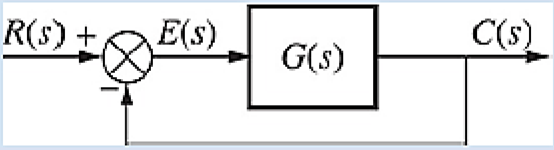


FIGURE 10.10 Closed-loop unity-feedback system

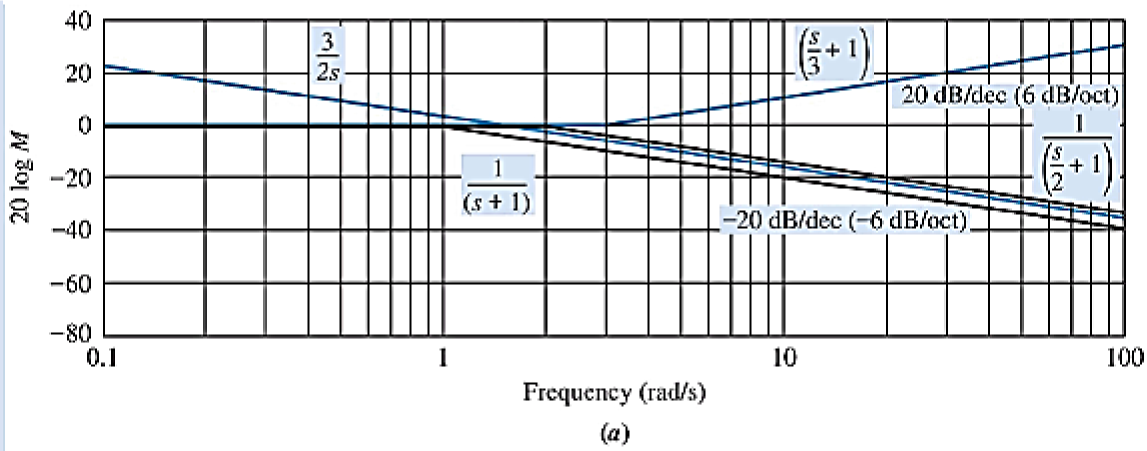
SOLUTION:

We will make a Bode plot for the open-loop function $G(s) = K(s + 3)/[s(s + 1)(s + 2)]$. The Bode plot is the sum of the Bode plots for each first order term. Thus, it is convenient to use the normalized plot for each of these terms so that the low-frequency asymptote of each term, except the pole at the origin, is at 0 dB, making it easier to add the components of the Bode plot. We rewrite $G(s)$ showing each term normalized to a low frequency gain of unity. Hence,

$$G(s) = \frac{\frac{3}{2}K\left(\frac{s}{3}+1\right)}{s(s+1)\left(\frac{s}{2}+1\right)} \qquad 9.36$$

Now determine that the break frequencies are at 1, 2, and 3. The magnitude plot should begin a decade below the lowest break frequency and extend a decade above the highest break frequency. Hence, we choose 0.1 radian to 100 radians, or three decades, as the extent of our plot.

At $\omega = 0.1$, the low-frequency value of the function is found from Eq. (9.36) using the low-frequency values for all of the $[(s/a) + 1]$ terms (i.e., $s = 0$) and the actual value for the s term in the denominator. Thus, $G(j0.1) \approx 3/2K/0.1 = 15K$. The effect of K is to move the magnitude curve up (increasing K) or down (decreasing K) by the amount of $20 \log K$. K has no effect upon the phase curve. If we choose $K = 1$, the magnitude plot can be denormalized later for any value of K that is calculated or known.



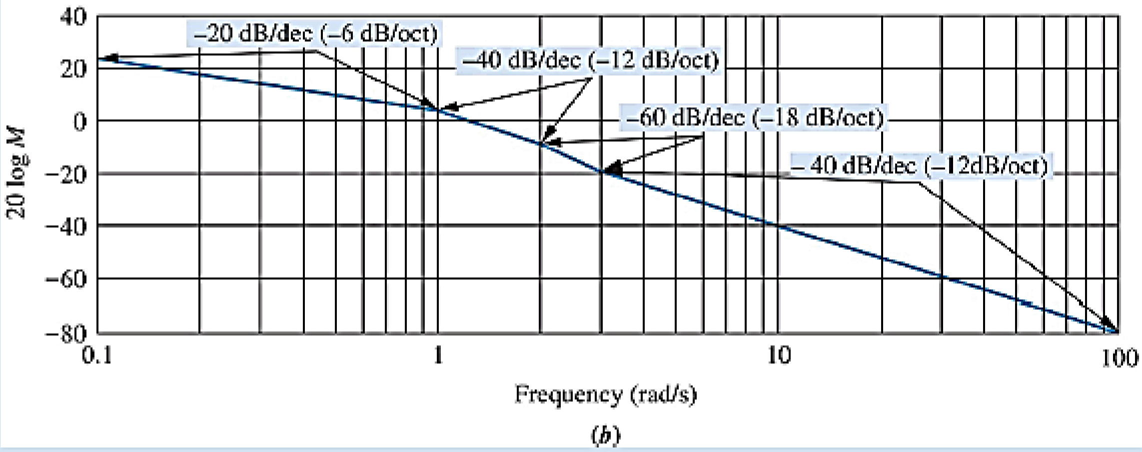


FIGURE 9.11 Bode log-magnitude plot for Example 9.2: a. components; b. composite

Figure 9.11(*a*) shows each component of the Bode log-magnitude frequency response. The Bode magnitude plot for $K = 1$ starts at $\omega = 0.1$ with a value of 20 log 15 = 23.52 dB, Summing the components yields the composite plot shown in Figure 10.11(*b*).

The results are summarized in Table 9.2, which can be used to obtain the slopes.

Bode magnitude plot: slope contribution from each pole and zero in Example 10.2

	Frequency (rad/s)			
Description	**0.1 (Start: Pole at 0)**	**1 (Start: Pole at −1)**	**2 (Start: Pole at −2)**	**3 (Start: Zero at −3)**
Pole at 0	−20	−20	−20	−20
Pole at −1	0	−20	−20	−20
Pole at −2	0	0	−20	−20
Zero at −3	0	0	0	20
Total slope (dB/dec)	−20	−40	−60	−40

Phase is handled similarly. However, the existence of breaks, a decade below and a decade above the break frequency, requires a little more bookkeeping. Table 9.3 shows the starting and stopping frequencies of the 45°/decade slope for each of the poles and zeros. For example, reading across for the pole at −2, we see that the −45° slope starts at a frequency of 0.2 and ends at 20. Filling in the rows for each pole and then summing the columns yields the slope portrait of the resulting phase plot. Looking at the row marked *Total slope*, we see that the phase plot will have a slope of −45°/decade from a frequency of 0.1 to 0.2. The slope will then increase to −90°/decade from 0.2 to 0.3. The slope will return to −45°/decade from 0.3 to 10 rad/s. A slope of 0 ensues from 10 to 20 rad/s, followed by a slope of +45°/decade from 20 to 30 rad/s. Finally, from 30 rad/s to infinity, the slope is 0°/decade.

Bode phase plot: slope contribution from each pole and zero in Example 10.2

	Frequency (rad/s)					
Description	**0.1 (Start: Pole at −1)**	**0.2 (Start: Pole at −2)**	**0.3 (Start: Pole at −3)**	**0 (End: Pole at −1)**	**20 (End: Pole at −2)**	**30 (End: Zero at −3)**
Pole at −1	−45	−45	−45	0		
Pole at −2		−45	−45	−45	0	
Zero at −3			45	45	45	0
Total slope (deg/dec)	−45	−90	−45	0	45	0

The resulting component and composite phase plots are shown in Figure 10.12. Since the pole at the origin yields a constant −90° phase shift, the plot begins at −90° and follows the slope portrait just described.

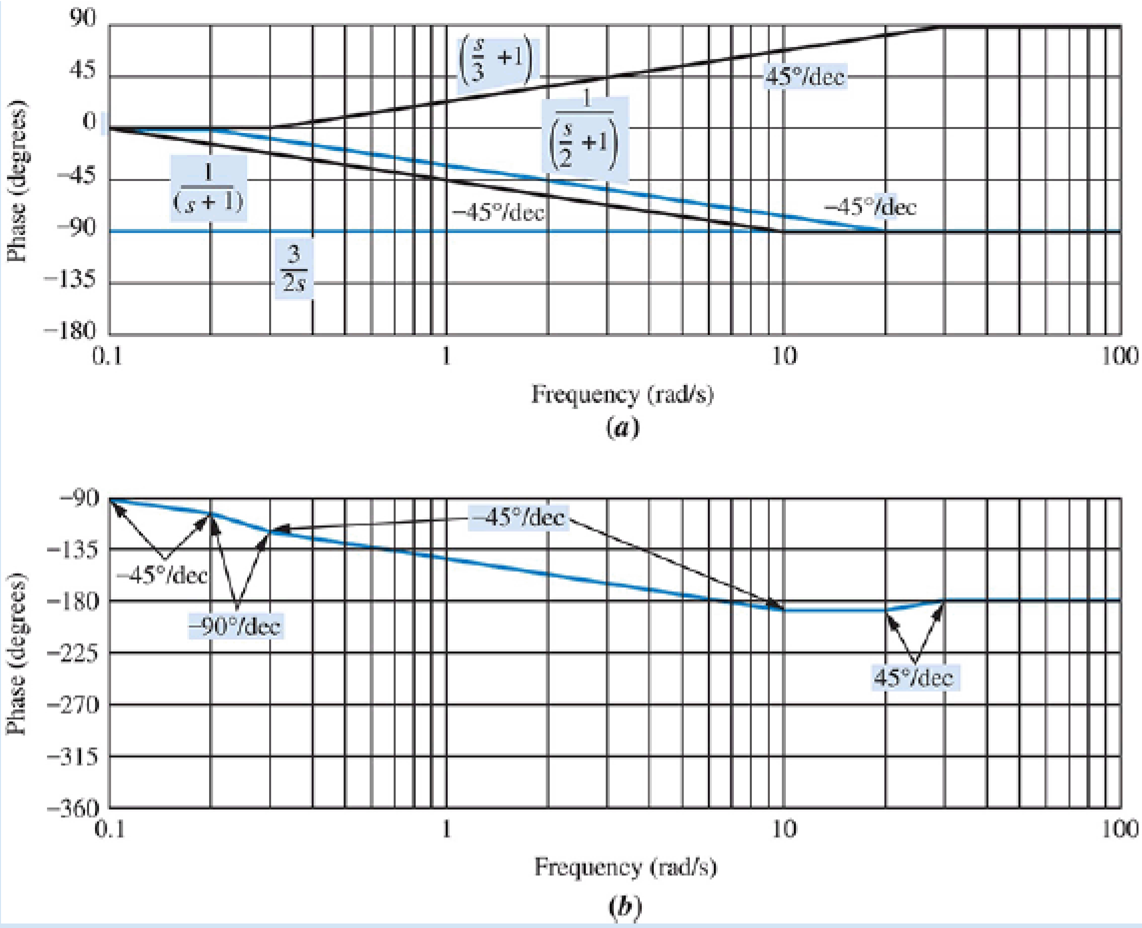


FIGURE 9.12 Bode phase plot for Example 9.3: a. components; b. composite

10.7 Stability, Gain Margin, and Phase Margin via Bode Plots

In this section, we determine stability, gain and phase margins, and the range of gain required for stability. All of these topics were covered previously in this chapter, using Nyquist diagrams as the tool. Now we use Bode plots to determine these characteristics. Bode plots are subsets of the complete Nyquist diagram but in another form. They are a viable alternative to Nyquist plots, since they are easily drawn without the aid of the computational devices or long calculations required for the Nyquist diagram and root locus. You should remember that all calculations applied to stability were derived from and based upon the Nyquist stability criterion. The Bode plots are an alternate way of visualizing and implementing the theoretical concepts.

10.7.1 Determining Stability

Let us look at an example and determine the stability of a system, implementing the Nyquist stability criterion using Bode plots. We will draw a Bode log-magnitude plot and then determine the value of gain that ensures that the magnitude is less than 0 dB (unity gain) at that frequency where the phase is ±180°.

Example 10.4 Range of Gain for Stability via Bode Plots

PROBLEM:

Use Bode plots to determine the range of K when $G(s) = K/[(s + 2)(s + 4)(s + 5)]$ and H(s)=1.

SOLUTION:

Since this system has all of its open-loop poles in the left half-plane, the open-loop system is stable. the closed-loop system will be stable if the frequency response has a gain less than unity when the phase is 180°. Begin by sketching the Bode magnitude and phase diagrams shown in Figure 10.10.

In Section 9.5, we summed normalized plots of each factor of $G(s)$ to create the Bode plot. We saw that at each break frequency, the slope of the resultant Bode plot changed by an amount equal to the new slope that was added. Table 10.1 demonstrates this observation. In this example, we use this fact to draw the Bode plots faster by avoiding the sketching of the response of each term.

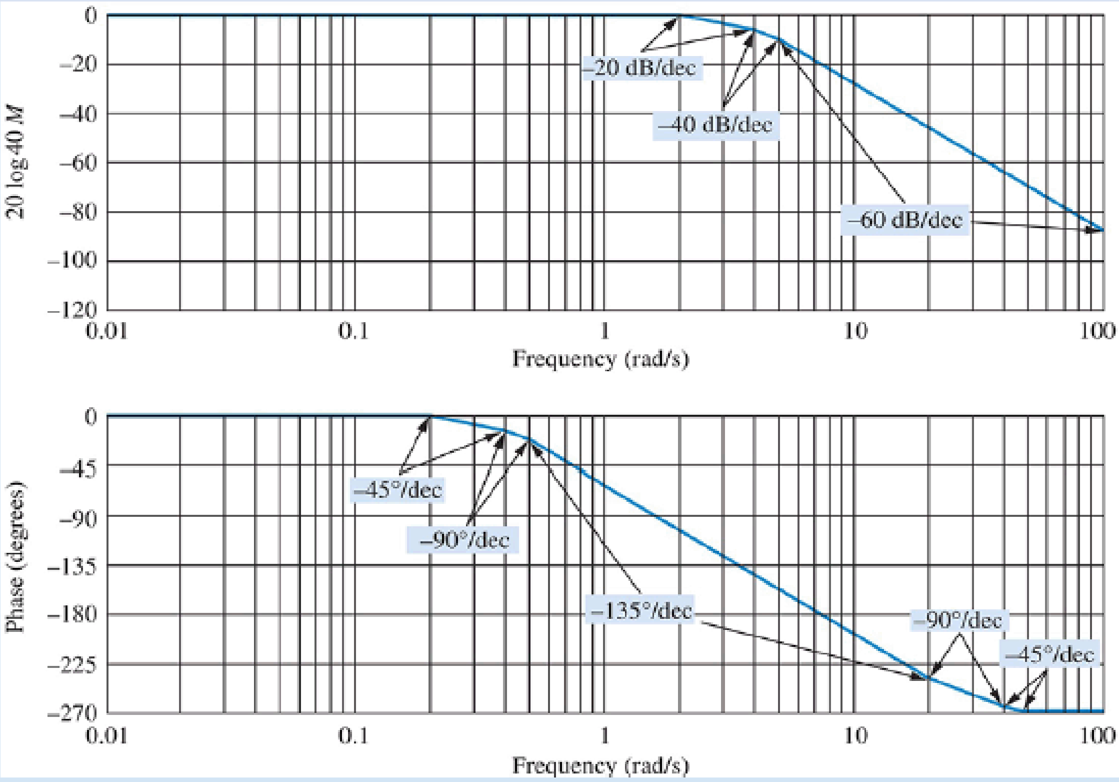


FIGURE 10.10 Bode log-magnitude and phase diagrams for the system of Example 10.4

The low-frequency gain of $G(s)H(s)$ is found by setting s to zero. Thus, the Bode magnitude plot starts at $K/40$. For convenience, let $K = 40$ so that the log-magnitude plot starts at 0 dB. At each break frequency, 2, 4, and 5, a 20-dB/decade increase in negative slope is drawn, yielding the log magnitude plot shown in Figure 10.10. The phase diagram begins at 0° until a decade below the first break frequency of 2 rad/s. At 0.2 rad/s, the curve decreases at a rate of −45°/decade, decreasing an additional 45°/decade at each subsequent frequency (0.4 and 0.5 rad/s) a decade below each break. At a decade above each break frequency, the slopes are reduced by 45°/decade at each frequency.

The Nyquist criterion for this example tells us that we want zero encirclements of −1 for stability. Thus, we recognize that the Bode log magnitude plot must be less than unity when the Bode phase plot is 180°.

Accordingly, we see that at a frequency of 7 rad/s, when the phase plot is −180°, the magnitude plot is −20 dB. Therefore, an increase in gain of +20 dB is possible before the system becomes unstable. Since the gain plot was scaled for a gain of 40, +20 dB (a gain of 10) represents the required increase in gain above 40. Hence, the gain for instability is 40 × 10 = 400. The final result is $0 < K < 400$ for stability. This result, obtained by approximating the frequency response by Bode asymptotes, can be compared to the result obtained from the actual frequency response, which yields a gain of 378 at a frequency of 6.16 rad/s.

Evaluating Gain and Phase Margins

Next, we show how to evaluate the gain and phase margins using Bode plots (Figure 10.11). The gain margin is found using the phase plot to find the frequency, ω_{GM} , where the phase angle is 180°. At this frequency, we look at the magnitude plot to determine the gain margin, *GM*, which is the gain required to raise the magnitude curve to 0 dB. To illustrate, in the previous example with $K = 40$, the gain margin was found to be 20 dB.

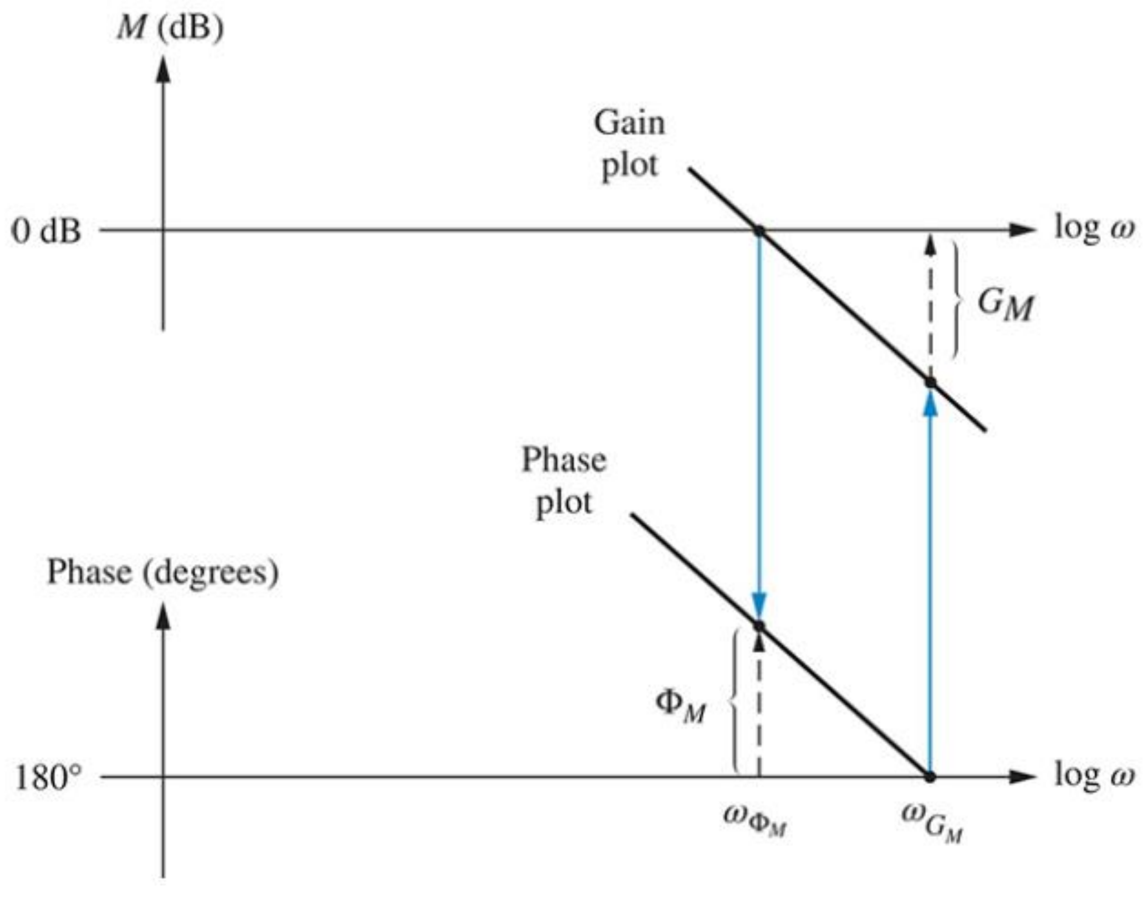


FIGURE 10.11 Gain and phase margins on the Bode diagrams

The phase margin is found using the magnitude curve to find the frequency, $\omega_{\Phi M}$, where the gain is 0 dB. On the phase curve at that frequency, the phase margin, ΦM, is the difference between the phase value and 180°.

Example 10.5 Gain and Phase Margins from Bode Plots

PROBLEM:

If $K = 200$ in the system of Example 10.4, find the gain margin and the phase margin.

SOLUTION:

The Bode plot in Figure 10.10 is scaled to a gain of 40. If $K = 200$ (five times as great), the magnitude plot would be 20 log 5 = 13.98 dB higher. To find the gain margin, look at the phase plot and find the frequency where the phase is 180°. At this frequency, determine from the magnitude plot how much the gain can be increased before reaching 0 dB. In Figure 10.10, the phase angle is 180° at approximately 7 rad/s. On the magnitude plot, the gain is −20 + 13.98 = − 6.02 dB. Thus, the gain margin is 6.02 dB.

To find the phase margin, we look on the magnitude plot for the frequency where the gain is 0 dB. At this frequency, we look on the phase plot to find the difference between the phase and 180°. This difference is the phase margin. Again, remembering that the magnitude plot of Figure 10.36 is 13.98 dB lower than the actual plot, the 0 dB crossing (−13.98 dB for the normalized plot shown in Figure 10.36) occurs at 5.5 rad/s. At this frequency the phase angle is −165°. Thus, the phase margin is −165° − (−180°) = 15°