Tikrit university

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Department of Electrical Engineering

Fourth Class- Semester-2

Control Engineering

Chapter 9

Lecture 3

Frequency response

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Using time domain characteristics it is possible to obtain an idea about the performance of a given system. If the order of the system increases, it is very difficult to analyse the system by the time domain approach. Also in design problem, there is no single method to determine the design solutions which obeys the various time domain specifications such as rise time, maximum overshoot, etc.

Frequency domain analysis is another analytic method that can be used for analysis and design. During 1930s, Bode and Nyquist proposed methods to analyse the stability of a system. The root locus method (i.e., time domain method) was developed during 1940s. The frequency response method was developed earlier than the time domain method. The aim of this chapter is to discuss the basics of the frequency domain analysis of systems.

9.1 Frequency Response

It is defined as the steady-state response of a system due to a sinusoidal input. Figure 9.1 shows a linear system with a sinusoidal input. Let us study its output and derive the relation between its input and output.

Fig. 9.1 A linear system with sinusoidal input

In Fig. 9.1 x(t) is the sinusoidal input function, y(t) is the output function, and g(t) is the transfer function. Let X(s), U(s) and G(s) be the Laplace transforms of x(t), y(t) and g(t), respectively. G(s) is the transfer function of the system and it is defined for the zero initial condition. G(s) is given by

$$G(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+a)(s+b)(s+c)\cdots} \qquad 9.1$$

In Eq. (9.1), N(s) is a polynomial of s and the poles of G(s) are at -a , -b, -c, etc.

Again $$G(s) = \frac{Y(s)}{X(s)} \qquad 9.2$$

Using Eq. (9.1) and (9.2), we have

$$Y(s) = \frac{N(s)X(s)}{(s+a)(s+b)(s+c)\cdots} \qquad 9.3$$

Let x(t) = A sin m be the input of the system. The Laplace transform of x(s) is given by

$$X(s) = \frac{A\omega}{s^2+\omega^2} \qquad 9.4$$

Using Eq. (9.4), Eq. (9.3) can be written as follows:

$$Y(s) = \left[\frac{N(s)}{(s+a)(s+b)(s+c)\cdots}\right]\left(\frac{A\omega}{s^2+\omega^2}\right)$$

$$= \frac{A_1}{s+a} + \frac{A_2}{s+b} + \frac{A_3}{s+c} + \ldots + \frac{B_1}{s+j\omega} + \frac{B_2}{s-j\omega}$$

$$y(t) = A_1 e^{-at} + A_2 e^{-bt} + A_3 e^{-ct} + \cdots + B_1 e^{-j\omega t} + B_2 e^{+j\omega t} \qquad 9.5$$

The terms with coefficients A_1 to A_n will decay to zero as time tends to infinity, because these terms contribute to decaying exponents. Therefore, these terms constitute transient response. The terms with coefficients *B1* and *B2* will contribute steady-state response. Therefore, the steady-state response of Eq. (9.5) is given by

$$y_{ss}(t) = B_1 e^{-j\omega t} + B_2 e^{j\omega t} \qquad 9.6$$

$$B_1 = \left[G(s)\frac{A\omega}{(s^2-\omega^2)}(s+j\omega)\right]_{s=-j\omega} = -\frac{A}{2j}G(-j\omega) \qquad 9.7$$

$$B_2 = \left[G(s)\frac{A\omega}{(s^2-\omega^2)}(s-j\omega)\right]_{s=j\omega} = \frac{A}{2j}G(j\omega) \qquad 9.8$$

G(jw) is a complex quantity and we can write G(jw) as follows:

$$G(j\omega) = |G(j\omega)|e^{j\phi} \qquad 9.9$$

where |G(jω) | represents the magnitude and f represents the angle of G(jw) Using Eqs. (9.7) and (9.8), we have Eq. (9.6):

$$y_{ss}(t) = -\frac{A}{2j}|G(j\omega)|e^{-j\phi}e^{-j\omega t} + \frac{A}{2j}|G(j\omega)|e^{j\phi}e^{j\omega t}$$

$$= A|G(j\omega)|\left\{\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j}\right\}$$

$$= A\,|G(j\omega)|\sin(\omega t+\phi)$$

or,

$$y_{ss}(t) = |Y(j\omega)| < \phi \qquad 9.10$$

where

$$|Y(j\omega)| = A|G(j\omega)| \qquad 9.11$$

From Eq. (9.11)

$$|Y(j\omega)| = |G(j\omega)|\,|X(j\omega)| \qquad 9.12$$

or

$$|G(j\omega)| = \frac{|Y(j\omega)|}{|X(j\omega)|} \qquad 9.13$$

and $$\angle|Y(j\omega)| = \phi = \phi + 0°$$ 9.14

or $$\angle|Y(j\omega)| = \angle|G(j\omega)| + \angle|X(j\omega)|$$ 9.14a

or $$\angle|G(j\omega)| = \angle|Y(j\omega)| - \angle|X(j\omega)|$$ 9.14b

Therefore, the frequency of sinusoidal output is the same as the input and there is merely change in the amplitude and phase of the output with respect to input.

9.2 Methods in Frequency Response

The magnitude function and phase function are the two plots in the frequency response characteristics. The angular frequency (ω) is the common variable for both these plots. The following curves are commonly used in frequency response analysis:

(i) Polar plot: The plot of the magnitude versus phase angle in polar coordinate where ω is varied from 0 to ∞ is called polar plot.

(ii) Bode plot: The plots of both (i) magnitude and (ii) phase versus frequency ω in logarithmic values are called Bode plots.

(iii) Magnitude versus plot: The magnitude versus phase plot on rectangular coordinate as varying co is known as magnitude versus phase plot.

When plotting separate magnitude and phase plots, the magnitude curve can be plotted in decibels (dB) vs. log ω, where dB = 20 log M. The phase curve is plotted as phase angle vs. log ω

The following example demonstrates how to obtain an analytical expression for frequency response and make a plot of the result.

Example 9.1

Find the analytical expression for the magnitude frequency response and the phase frequency response for a system G(s) = 1/(s + 2). Also, plot both the separate magnitude and phase diagrams and the polar plot.

SOLUTION:

First substitute $s = j\omega$ in the system function and obtain $G(j\omega) = 1/(j\omega + 2)$ $= (2 - j\omega)/(\omega^2 + 4)$. The magnitude of this complex number,

$|G(j\omega)| = M(\omega) = 1/\sqrt{\omega^2 + 4}$),

is the magnitude frequency response. The phase angle of $G(j\omega)$,

$\phi(\omega) = -\tan^{-1}(\omega/2)$, is the phase frequency response.

$G(j\omega)$ can be plotted in two ways: (1) in separate magnitude and phase plots and (2) in a polar plot. Figure E.1a shows separate magnitude and phase diagrams, where the magnitude diagram is

$20 \log M(\omega) = 20 \log (1/\sqrt{\omega^2 + 4})$ vs. $\log \omega$, and the phase diagram is

$\phi(\omega) = -\tan - 1\ (\omega/2)$ vs. $\log \omega$. The polar plot, shown in Figure E1b, is a plot of $M(\omega) < \phi(\omega) = 1/\sqrt{\omega^2 + 4} < -\tan^{-1}(\omega/2)$ for different ω

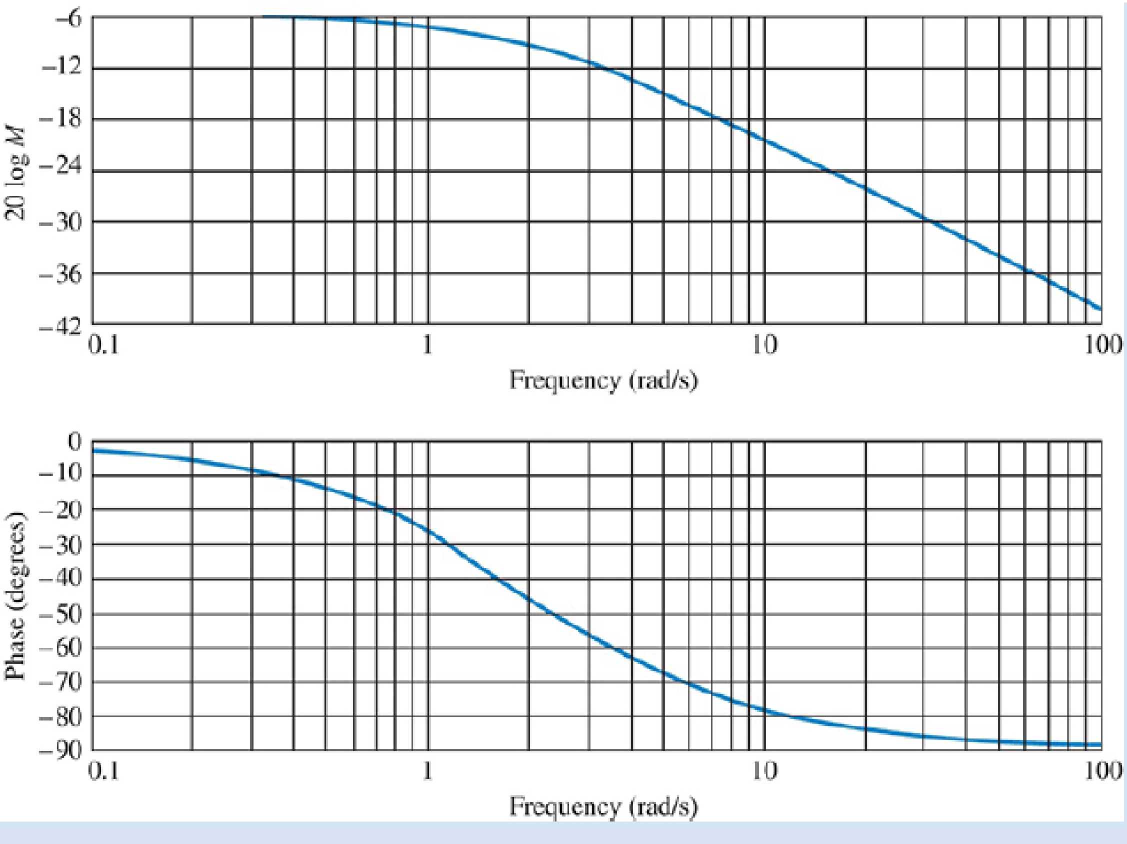


Fig E1 Frequency response plots for G(s) = 1/(s + 2): separate magnitude and phase diagrams

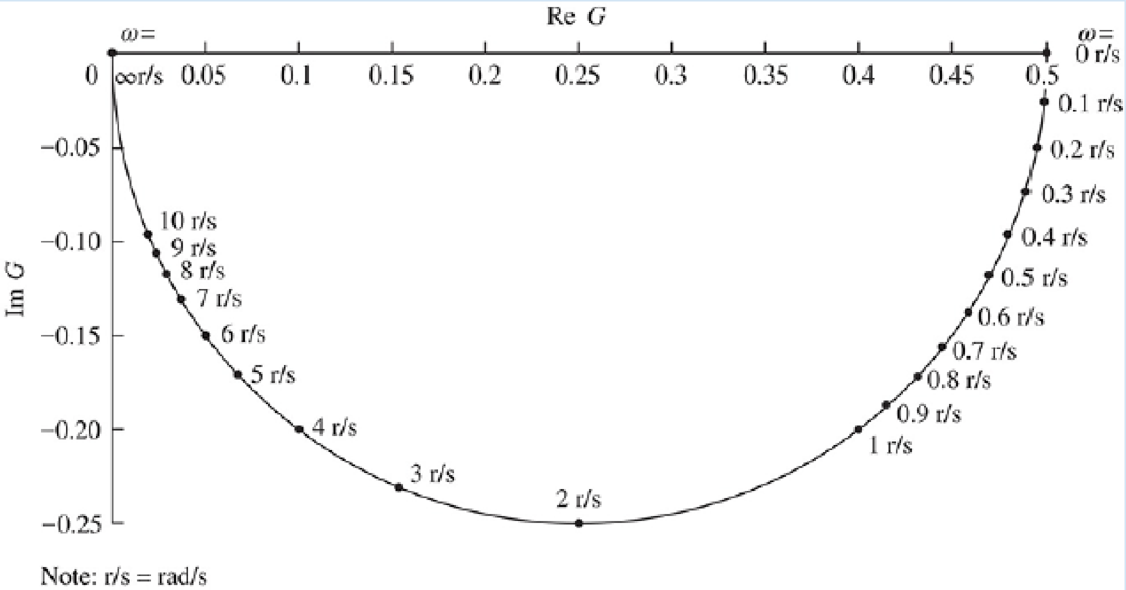


Fig. E2 Frequency response plot for $G(s) = 1/(s + 2)$: polar plot

9.3 Definitions of Frequency Domain Specifications

Response peak (M_r): From Fig. 9.2, the closed-loop transfer function is given by

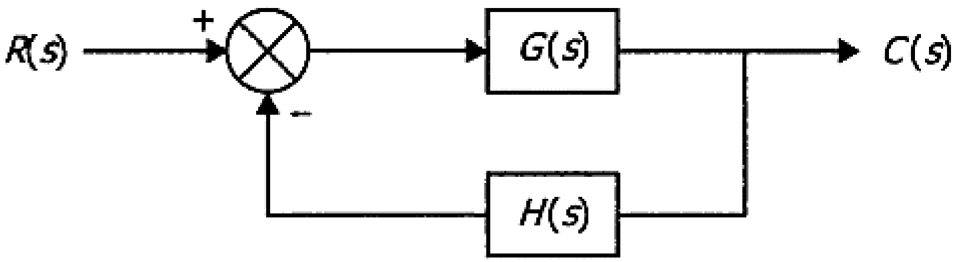


Fig. 9.2 Closed-loop system

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = M(s)$$

or $$M(j\omega) = \frac{G(j\omega)}{1+G(j\omega)H(j\omega)} \qquad 9.15$$

The maximum value of $M(j\omega)$ as ω varied is termed as resonance peak (M_r). The relative stability of the closed-loop system can be indicated by M_r. If M_r is large, there will be large maximum overshoot in a step response. Generally, M_r is between 1.1 and 1.5.

Resonance frequency: The frequency at which peak resonance M_r occurs is known as resonance frequency ω_r.

Bandwidth (*.BW):* The frequency at which the magnitude of M(jω) drops by 3 dB below a certain specified level, generally 0 dB level. If bandwidth is large, higher frequencies will pass through the system. This indicates a faster rise in time. Therefore, bandwidth gives an idea about transient response.

Cut off rate: The slope of the log magnitude curve near the cut-off frequency is known as cut off rate. The ability of the system to distinguish a signal from noise can be determined by cut-off rate.

Phase Margin: if Φ is the phase angle of a system at unity gain, the phase margin is given by (180 + Φ). The increment in the system angle that causes the system to be unstable is indicated by phase margin. It is a measure of relative stability.

Gain Crossover Frequency: The frequency (ω) at which gain is unity is known as gain crossover frequency.

Gain Margin. The increment in gain required to cause the system to become unstable is indicated by gain margin.

phase crossover frequency. The frequency (ω) at which the phase angle of the transfer function is -180° is known as phase crossover frequency.

9.4 Correlation Between Time and Frequency Domain Specifications for a Second-Order System

The relationship between frequency domain specifications and time domain specifications is important to get these ideas before designing the system. For a canonical second-order system

$$G(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\,\omega_n s + \omega_n^2}$$

and $$H(s) = 1$$

or $$G(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2}$$

$\therefore$ $$G(j\omega) = \frac{1}{1 + 2j\xi\left(\frac{\omega}{\omega_n}\right) - \left(\frac{\omega}{\omega_n}\right)^2} \qquad 9.16$$

Let $$u = \frac{\omega}{\omega_n}$$

$\therefore$ $$G(j\omega) = \frac{1}{1 + 2ju\xi - u^2} \qquad 9.17$$

or $$|G(u)| = M = \frac{1}{\sqrt{(1-u^2)^2 + (2u\xi)^2}} \qquad 9.18$$

and $$\angle G(j\omega) = -\tan^{-1}\frac{2u\xi}{1-u^2} \qquad 9.19$$

Putting $\left.\frac{dM}{du}\right|_{u=u_r} = 0$, the resonant frequency ω_r can be obtained.

Now $$\left.\frac{dM}{du}\right|_{u=u_r} = -\frac{1}{2}\,\frac{\left[2(1-u_r^2)(-2u_r) + 8\xi^2 u_r\right]}{\left[(1-u_r^2)^2 + (2u_r\xi)^2\right]^{3/2}} = 0$$

or $$4u_r - 8u_r\xi^2 - 4u_r^3 = 0$$

or $$4u_r(1 - 2\xi^2 - u_r^2) = 0$$

$\therefore$ $$u_r = 0$$

or $$1 - 2\xi^2 - 4u_r^2 = 0$$

Now $u_r = 0$ gives $\omega_r/\omega_n = 0$ or $\omega_r = 0$.

Again $$1 - 2\xi^2 - u_r^2 = 0$$

or $$u_r^2 = 1 - 2\xi^2 \text{ or } u_r = \sqrt{1-2\xi^2} \qquad 9.20$$

or $$\frac{\omega_r}{\omega_n} = \sqrt{1-2\xi^2}$$

or $$\omega_r = \omega_n\sqrt{1-2\xi^2} \qquad 9.21$$

$\therefore$ $$M_r = \left.G(j\omega)\right|_{u=u_r} = \frac{1}{\sqrt{(1-u_r^2)^2 + (2u_r\xi)^2}} = \frac{1}{\sqrt{(2\xi^2)^2 + 4\xi^2(1-2\xi^2)}}$$

$$= \frac{1}{\sqrt{4\xi^4 + 4\xi^2 - 8\xi^4}} = \frac{1}{\sqrt{4\xi^2 - 4\xi^4}} = \frac{1}{\sqrt{4\xi^2(1-\xi^2)}}$$

$\therefore$ $$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} \qquad 9.22$$

From Eq. (9.20), u_r is valid as long as $1-2\xi^2 > 0$, i.e., $\xi < 0.707$ and Eq. (9.22) shows that Mr is a function of ξ. From Eq. (12.19), we have

$$\phi_r = G(ju)\big|_{u=u_r} = -\tan^{-1}\frac{\sqrt{1-2\xi}}{\xi} \qquad 9.23$$

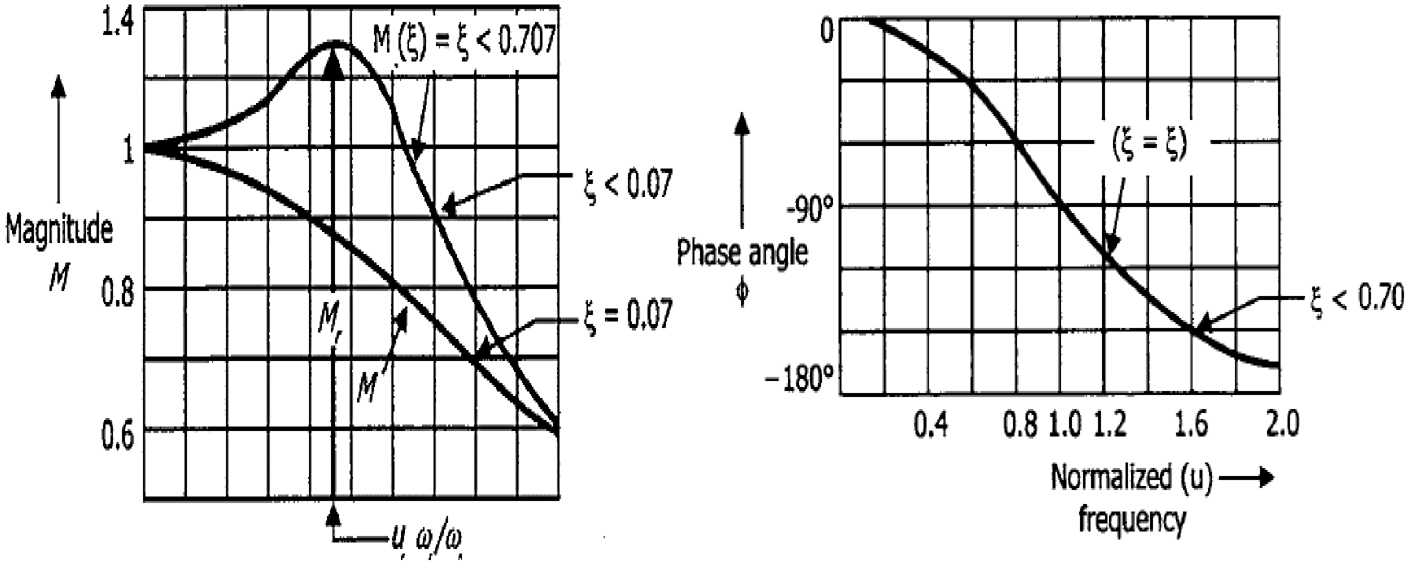
Equation (9.21) shows that ωr is a function of ξ and ωn From Eq. (9.21) and (9.22), the following conclusions can be drawn:

(i) *Mr* approaches infinity and ω_r approaches ω_n when ξ, approaches zero.

(ii) *Mr* is greater than unity and ω_r is less than ω_n for $0 < \xi < 0.707$.

(iii) There is no resonant peak and the maximum value of *Mr* is unity and $\omega r = 0$ when $\xi > 0.707$.

Figures 9.3(a) and 9.3(b) show the plot of magnitude *(M)* versus normalized frequency (u) and phase angle (Φ) versus normalized frequency (u), respectively.



(a) Frequency response magnitude characteristics (b) Frequency response characteristic

Fig. 12.3 Frequency Response

To determine cut off rate and BW let us consider the following figures. Figure 9.4 shows the plot of Mr versus ω known as the typical magnification curve of a feedback control system. Figure 9.5 shows the plot of bandwidth versus damping factor.

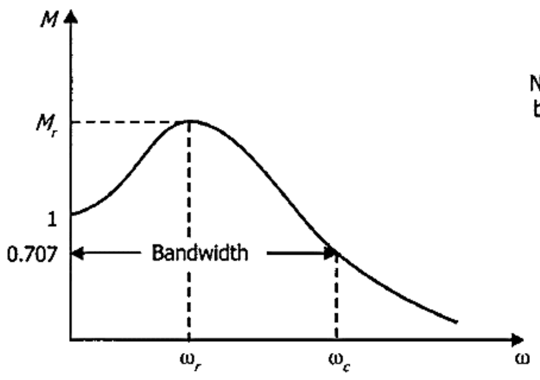


Fig. 9.4 Plot of Mr vs ω

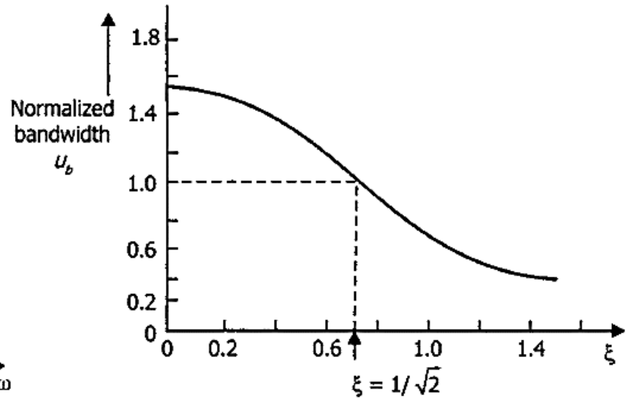


Fig. 9.5 Bandwidth vs ξ

At cut-off frequency (ω_c) the value of M is 0.707. Generally, control systems are low pass-filters and at zero frequency M = 1. Therefore, bandwidth becomes equal to ω_c. Taking M = 0.707 and u = u_c in Eq. (9.18), we get

$$0.707 = \frac{1}{\left[(1-u_c^2)^2 + (2u_c\xi)^2\right]^{\frac{1}{2}}}$$

where $$u_c = \frac{\omega_c}{\omega_n}$$

or $$(1-u_c^2)^2 + (2u_c\xi)^2 = 2 \qquad 9.24$$

Solving Eq. (9.24), we have

$$u_c^2 = (1-2\xi^2) \pm \sqrt{4\xi^4 - 4\xi^2 + 2}$$

or $$\omega_c = \omega_n \left\{(1-2\xi^2) + \sqrt{4\xi^4 - 4\xi^2 + 2}\right\}^{1/2}$$

$$\mathrm{BW} = \omega_c \qquad 9.25$$

Positive sign is considered because BW is a positive quantity. Therefore, Eq. (9.25) shows that BW is a function of ω_n and ξ.

$$\frac{\omega_d}{\omega_r} = \frac{1-\xi^2}{\sqrt{1-2\xi^2}} \qquad 9.26$$

Example 9.2 Determine the frequency domain specifications for a second-order system with unity feedback

and G(s) = 225/s (s+ 6)].

Solution

The transfer function *M(s)* is given by

$$M(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{225}{s(s+6)}}{1+\frac{225}{s(s+6)}} = \frac{225}{s^2+6s+225}$$

Comparing it with

$$\frac{\omega_n^2}{s^2+2\xi\omega_n+\omega_n^2}$$

we get

$$\omega_n^2 = 225 \text{ or } \omega_n = 15 \text{ rad/sec}$$

and $2\xi\omega_n = 6$

or $$\xi = \frac{6}{2\omega_n} = \frac{6}{2\times 15} = 0.2$$

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{1}{2\times 0.2\sqrt{1-(0.2)^2}} = 2.53$$

Again $$\omega_r = \omega_n\sqrt{1-2\xi^2} = 15\sqrt{1-2\times(0.2)^2} = 14.39 \text{ rad/sec}$$

Again $$\omega_c = \omega_n\left\{(1-2\xi^2)+\sqrt{4\xi^4-4\xi^2+2}\right\}^{1/2} = 15\times(0.92+1.359)^{1/2}$$

$$= 22.64 \text{ rad/sec}$$

Therefore, $$BW = \omega_c = 22.64 \text{ rad/sec}$$