Tikrit university

Collage of Engineering Shirqat

Department of Electrical Engineering

Fourth Class- Semester-2

Control Engineering

Chapter 8

Lecture 2

Root Locus Techniques

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8.5 Refining the Sketch

The rules covered in the previous section permit us to sketch a root locus rapidly. If we want more detail, we must be able to accurately find important points on the root locus along with their associated gain. Points on the real axis where the root locus enters or leaves the complex plane—real-axis breakaway and break-in points—and the jω-axis crossings are candidates. We can also derive a better picture of the root locus by finding the angles of departure and arrival from complex poles and zeros, respectively.

In this section, we discuss the calculations required to obtain specific points on the root locus. Some of these calculations can be made using the basic root locus relationship that the sum of the zero angles minus the sum of the pole angles equals an odd multiple of 180°, and the gain at a point on the root locus is found as the ratio of (1) the product of pole lengths drawn to that point to (2) the product of zero lengths drawn to that point.

We now discuss how to refine our root locus sketch by calculating real axis breakaway and break-in points, jω-axis crossings, angles of departure from complex poles, and angles of arrival to complex zeros. We conclude by showing how to find accurately any point on the root locus and calculate the gain.

8.5.1 Real-Axis Breakaway and Break-In Points

Numerous root loci appear to break away from the real axis as the system poles move from the real axis to the complex plane. At other times the loci appear to return to the real axis as a pair of complex poles becomes real. We illustrate this in Figure 8.13. This locus is sketched using the first four rules: (1) number of branches, (2) symmetry, (3) real-axis segments, and (4) starting and ending points. The figure shows a root locus leaving the real axis between −1 and −2 and returning to the real axis between +3 and +5. The point where the locus leaves the real axis, $-\sigma_1$, is called the breakaway point, and the point where the locus returns to the real axis, σ_2, is called the break-in point.

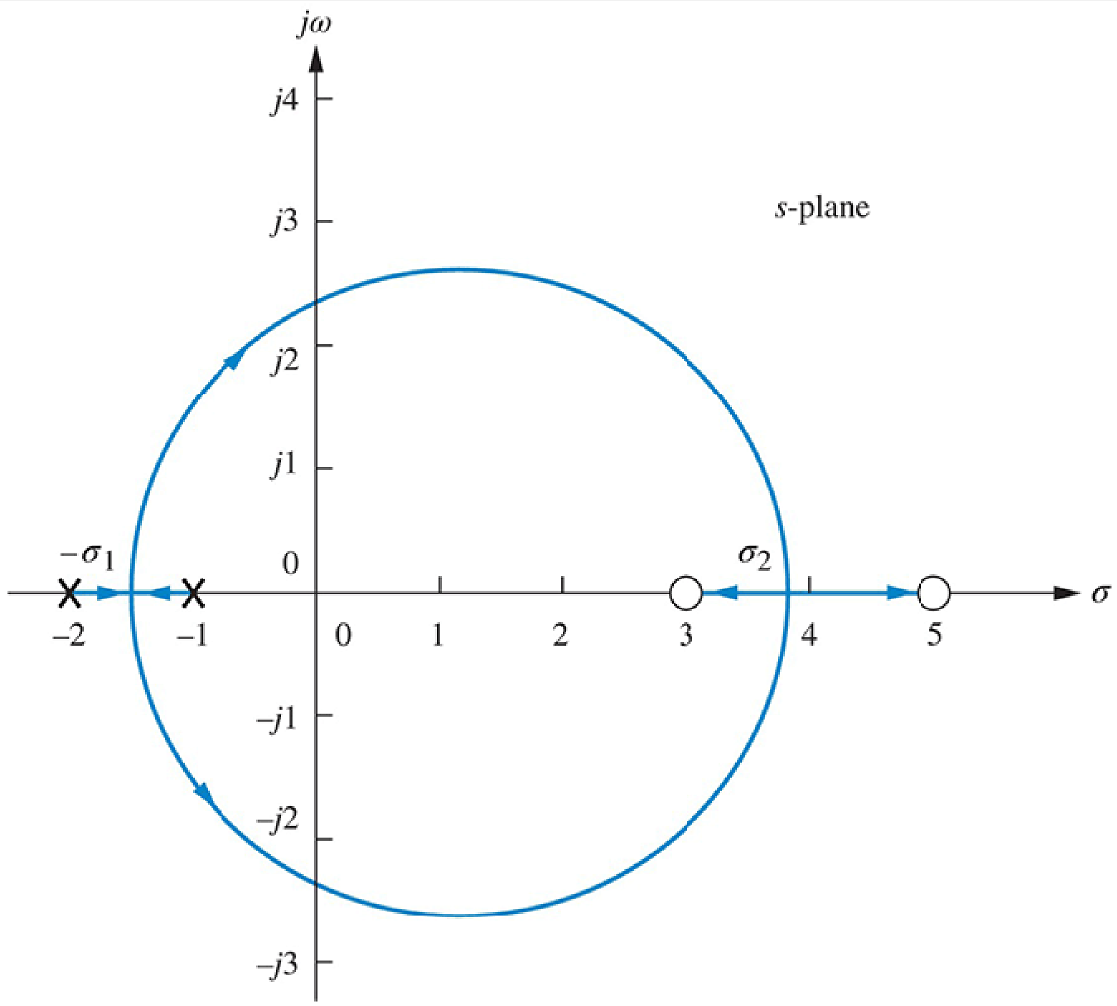


FIGURE 8.13 Root locus example showing real-axis breakaway (−σ1) and break-in points (σ2)

We now show how to find the breakaway and break-in points. As the two closed-loop poles, which are at −1 and −2 when K = 0, move toward each other, the gain increases from a value of zero. We conclude that the gain must be maximum along the real axis at the point where the breakaway occurs, somewhere between −1 and −2. Naturally, the gain increases above this value as the poles move into the complex plane. We conclude that the breakaway point occurs at a point of maximum gain on the real axis between the open-loop poles.

Now let us turn our attention to the break-in point somewhere between +3 and +5 on the real axis. When the closed-loop complex pair returns to the real axis, the gain will continue to increase to infinity as the closed-loop poles move toward the open-loop zeros.

It must be true, then, that the gain at the break-in point is the minimum gain found along the real axis between the two zeros.

The sketch in Figure 8.14 shows the variation of real-axis gain. The breakaway point is found at the maximum gain between −1 and −2, and the break-in point is found at the minimum gain between +3 and +5.

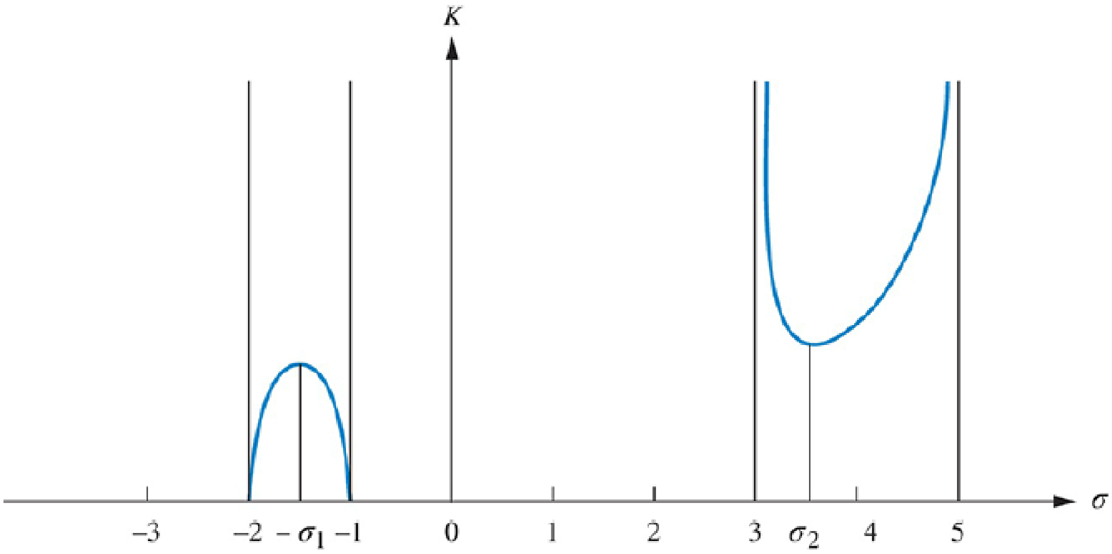


FIGURE 8.14 Variation of gain along the real axis for the root locus of Figure 8.13

There are three methods for finding the points at which the root locus breaks away from and breaks into the real axis.

The first method is to maximize and minimize the gain, K, using differential calculus. For all points on the root locus, Eq. (8.13) yields

$$K = -\frac{1}{G(s)H(s)} \quad (8.31)$$

For points along the real-axis segment of the root locus where breakaway and break-in points could exist, s = σ. Hence, along the real axis Eq. (8.31) becomes

$$K = -\frac{1}{G(\sigma)H(\sigma)} \quad (8.32)$$

This equation then represents a curve of K versus σ similar to that shown in Figure 8.14. Hence, if we differentiate Eq. (8.32) with respect to σ and set the derivative equal to zero, we can find the points of maximum and minimum gain and hence the breakaway and break-in points. Let us demonstrate.

Steps to determine the break-away points

Step 1: Frame the characteristic equation 1 + G(s)H(s) = 0 of the system.

Step 2: Write K in terms of s, i.e., K =f(s).

Step 3: Derive dk/ds and put dk/ds = 0

Step 4: The roots of the equation dk/ds = 0 are the break-away points.

If the value of K is positive for any root of dk/ds = 0, the root(s) is (are) valid break-away/break-in

point(s).

Example 8.3

Find the breakaway and break-in points for the root locus of Figure 8.13, using differential calculus.

SOLUTION: Using the open-loop poles and zeros, we represent the open-loop system whose root locus is shown in Figure 8.13 as follows:

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2-8s+15)}{(s^2+3s+2)} \quad (8.33)$$

But for all points along the root locus, KG(s)H(s) = − 1, and along the real axis, s = σ. Hence,

$$\frac{K(\sigma^2-8\sigma+15)}{(\sigma^2+3\sigma+2)} = -1 \quad (8.34)$$

Solving for *K*, we find

$$K = \frac{-(\sigma^2 + 3\sigma + 2)}{(\sigma^2 - 8\sigma + 15)} \quad (8.35)$$

Differentiating K with respect to σ and setting the derivative equal to zero yields

$$\frac{dK}{d\sigma} = \frac{(11\sigma^2 - 26\sigma - 61)}{(\sigma^2 - 8\sigma + 15)^2} = 0 \quad (8.36)$$

Solving for σ, we find σ = − 1.45 and 3.82, which are the breakaway and break-in points.

Example 8.4 For G(s)H(s) = K/[s(s + 1) (s + 3)], determine the coordinates of valid break-away/break-in

point(s).

Solution

Step 1: $1 + G(s)H(s) = 0$

or

$$1 + \frac{K}{s(s+2)(s+3)} = 0$$

or

$$s^3 + 5s^2 + 6s + K = 0$$

Step 2: $K = -s^3 - 5s^2 - 6s$

Step 3: $\frac{dK}{ds} = -3s^2 - 10s - 6 = 0$

Step 4: $3s^2 + 10s + 6 = 0$

∴ roots are $= \frac{-10 \pm \sqrt{10^2 - 4.3.6}}{2\times 3} = \frac{-10 \pm \sqrt{100-72}}{6} = \frac{-10 \pm \sqrt{28}}{6} = \frac{-10 \pm 5.29}{6}$

$= -0.785, -2.55$

Now

for $s = -0.785$

$K = -(-0.785)^3 - 5(-0.785)^2 - 6(-0.785) = 2.113$

Now

for $s = -2.55$,

$K = -(-2.55)^3 - 5(-2.55)^2 - 6(-2.55) = -0.631$

Thus for $s = -0.785$, K is positive and for $s = -2.55$, K is negative.

Therefore, $s = -0.785$ is a valid break-away point for the root locus.

The second method is a variation on the differential calculus method. Called the ***transition method***, it eliminates the step of differentiation. Breakaway and break-in points satisfy the relationship

$$\sum_{1}^{m} \frac{1}{\sigma+z_i} = \sum_{1}^{n} \frac{1}{\sigma+p_i} \qquad (8.37)$$

where zi and pi are the negative of the zero and pole values, respectively, of G(s)H(s). Solving Eq. (8.37) for σ, the real-axis values that minimize or maximize K, yields the breakaway and break-in points without differentiating. Let us look at an example.

Example 8.5

Repeat Example 8.3 without differentiating.

SOLUTION:

Using Eq. (8.37),

$$\frac{1}{\sigma-3}+\frac{1}{\sigma-5}=\frac{1}{\sigma+1}+\frac{1}{\sigma+2} \qquad (8.38)$$

Simplifying,

$$11\sigma^2-26\sigma-61=0 \qquad (8.39)$$

Hence, σ = − 1.45 and 3.82, which agrees with Example 8.3.

For the third method,

Simply use the program to search for the point of maximum gain between −1 and −2 and to search for the point of minimum gain between +3 and +5. Table 8.2 shows the results of the search. The locus leaves the axis at −1.45, the point of maximum gain between −1 and −2, and reenters the real axis at +3.8, the point of minimum gain between +3 and +5. These results are the same as those obtained using the first two methods.

TABLE 8.2

Data for breakaway and break-in points for the root locus of Figure 8.13

Real-axis value	Gain	Comment
−1.41	0.008557	
−1.42	0.008585	
−1.43	0.008605	
−1.44	0.008617	
−1.45	0.008623	← Max. gain : breakaway
−1.46	0.008622	
3.3	44.686	
3.4	37.125	
3.5	33.000	
3.6	30.667	
3.7	29.440	
3.8	29.000	← Min. gain : break-in
3.9	29.202	

8.5.2 The jω-Axis Crossings

We now further refine the root locus by finding the imaginary-axis crossings. The importance of the $j\omega$-axis crossings should be readily apparent. Looking at Figure 8.12, we see that the system's poles are in the left half-plane up to a particular value of gain. Above this value of gain, two of the closed-loop system's poles move into the right halfplane, signifying that the system is unstable. The $j\omega$-axis crossing is a point on the root locus that separates the stable operation of the system from the unstable operation. The value of ω at the axis crossing yields the frequency of oscillation, while the gain at the $j\omega$-axis crossing yields, for this example, the maximum positive gain for system stability. We should note here that other examples illustrate instability at small values of gain and stability at large values of gain. These systems have a root locus starting in the right half-plane (unstable at small values of gain) and ending in the left half-plane (stable for high values of gain). To find the $j\omega$-axis crossing, we can use the Routh–Hurwitz criterion, as follows: Forcing a row of zeros in the Routh table will yield the gain; going back one row to the even polynomial equation and solving for the roots yields the frequency at the imaginary-axis crossing.

To find the intersection of root locus with the imaginary

axis, the following procedures are followed.

Step 1: Construct the characteristic equation 1 + G(s)H(s) = 0.

Step 2: Develop Routh's array in terms of K.

Step 3: Find Kmar that creates one of the roots of Routh's array as a row of zeros.

Step 4: Frame auxiliary equation A(s) = 0 with the help of the coefficient of a row just above the row of zeros.

Step 5: The roots of the auxiliary equation A(s) = 0 for K = Kmar give the intersection points of the root locus with the imaginary axis.

Example 8.6 For G(s)H(s) = K/[s(s + 1)0 + 3)], find the point of the root locus with the jw-axis.

Solution

1 + *G(s)H(s)* = 0 , 1+ K/[s(s + 1)0 + 3)] $\Rightarrow s^3 + 6s^2 + 8s + K = 0$

Routh's array

s^3	1	8
s^2	6	K
s^1	$\frac{48-K}{6}$	0
s^0	K	

Now K = 48 and the row corresponding to s^1 becomes a row of zeros. The auxiliary equation with the help of the coefficients corresponding to s2 is given by

$6s^2 + K = 0$ or $6s^2 + 48 = 0$ or $s^2 = -8$ $s = \pm j2\sqrt{2}$

Therefore, $s = \pm j2\sqrt{2}$ are the points of the intersection of root locus with the imaginary axis.

8.5.3 Angles of Departure and Arrival

In this subsection, we further refine our sketch of the root locus by finding angles of departure and arrival from complex poles and zeros. Consider Figure 8.15, which shows the open-loop poles and zeros, some of which are complex. The root locus starts at the open-loop poles and ends at the open-loop zeros. In order to sketch the root locus more accurately, we want to calculate the root locus departure angle from the complex poles and the arrival angle to the complex zeros.

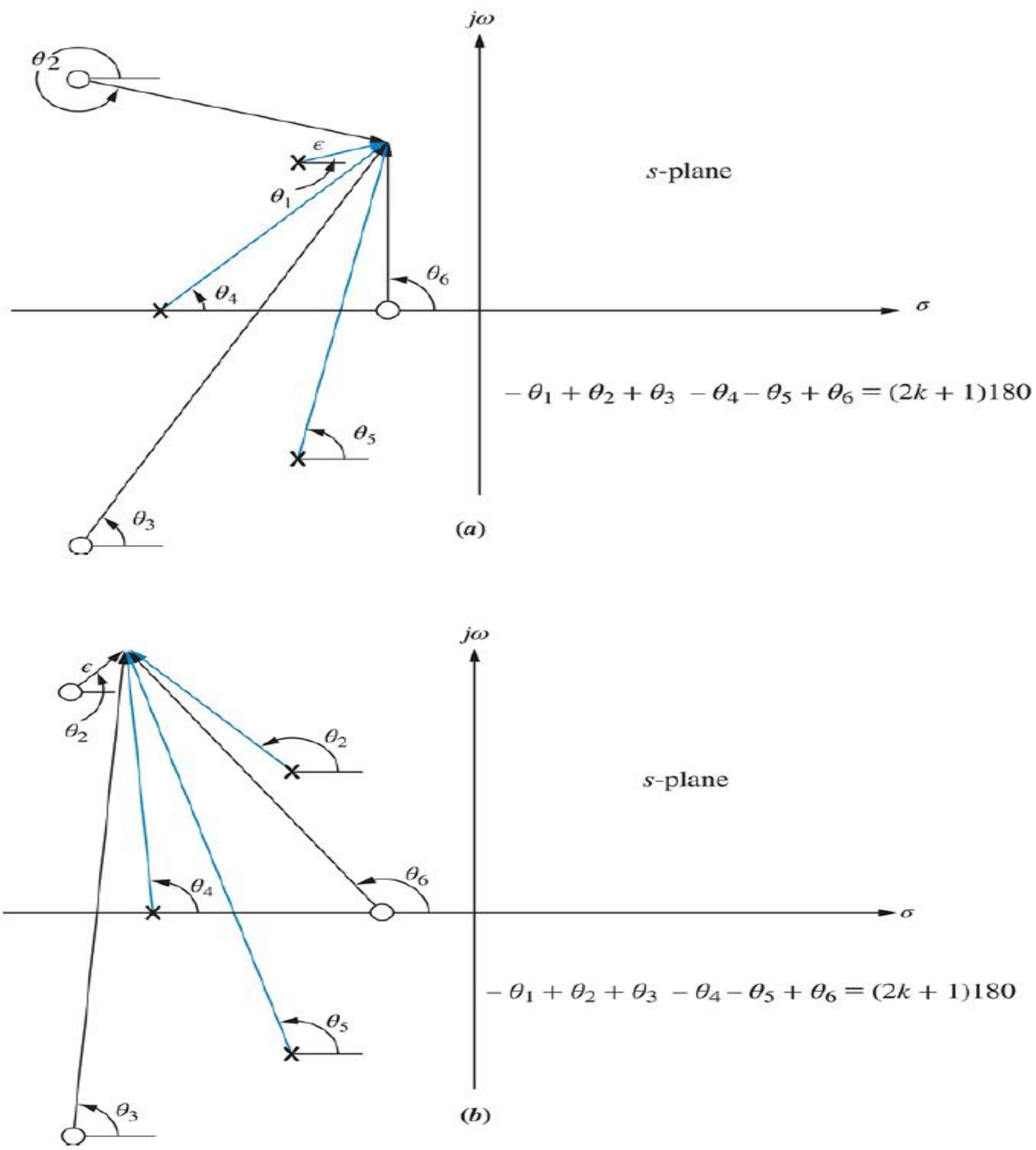


FIGURE 8.15 Open-loop poles and zeros and calculation of a. angle of departure; b. angle of arrival

The first method: If we assume a point on the root locus ε close to a complex pole, the sum of angles drawn from all finite poles and zeros to this point is an odd multiple of 180°. Except for the pole that is ε close to the point, we assume all angles drawn from all other poles and zeros are drawn directly to the pole that is near the point. Thus, the only unknown angle in the sum is the angle drawn from the pole that is ε close. We can solve for this unknown angle, which is also the angle of departure from this complex pole. Hence, from Figure 8.15(a),

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k+1)\,180^\circ \qquad (8.44a)$$

or

$$\theta_1 = \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 - (2k+1)\,180^\circ \qquad (8.44b)$$

If we assume a point on the root locus ε close to a complex zero, the sum of angles drawn from all finite poles and zeros to this point is an odd multiple of 180°. Except for the zero that is ε close to the point, we can assume all angles drawn from all other poles and zeros are drawn directly to the zero that is near the point. Thus, the only unknown angle in the sum is the angle drawn from the zero that is ε close. We can solve for this unknown angle, which is also the angle of arrival to this complex zero. Hence, from Figure 8.15(b),

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k+1)\,180^\circ \qquad (8.45a)$$

or

$$\theta_2 = \theta_1 - \theta_3 + \theta_4 + \theta_5 - \theta_6 + (2k+1)\,180^\circ \qquad (8.45b)$$

Example 8.7

Given the unity-feedback system of Figure 8.16, find the angle of departure from the complex poles and sketch the root locus.

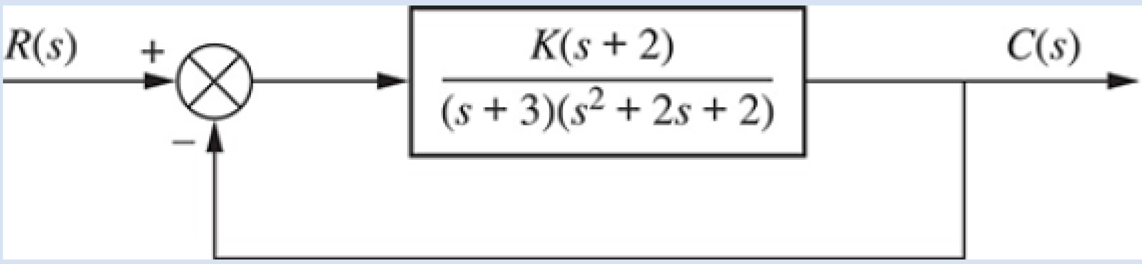


FIGURE 8.16 Unity-feedback system with complex poles

SOLUTION:

Using the poles and zeros of G(s) = (s + 2)/[(s + 3) (s2 + 2s + 2)] as plotted in Figure 8.17, we calculate the sum of angles drawn to a point ε close to the complex pole, −1 + j1, in the second quadrant. Thus,

$$-\theta_1 - \theta_2 + \theta_3 - \theta_4 = -\theta_1 - 90^\circ + \tan^{-1}\left(\frac{1}{1}\right) - \tan^{-1}\left(\frac{1}{2}\right) = 180^\circ$$

from which θ = − 251.6° = 108.4°. A sketch of the root locus is shown in Figure 8.17. Notice how the departure angle from the complex poles helps us to refine the shape.

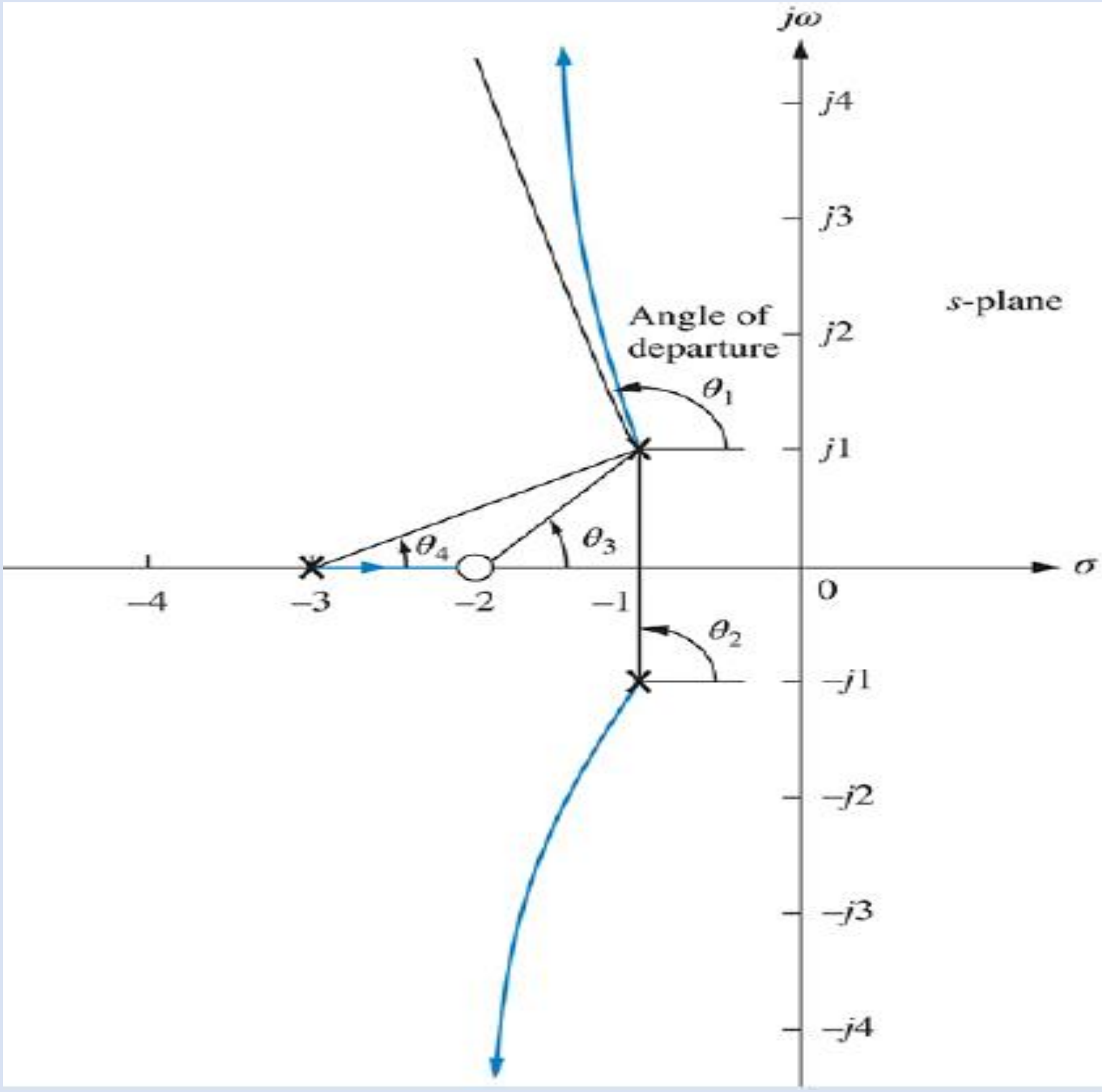


FIGURE 8.17 Root locus for system of Figure 8.16 showing angle of departure

The second method: Angle of departure/arrival: The root locus leaves from a complex pole and arrives at a complex zero. These two angles are known as angle of departure and angle of arrival, respectively. Angle of departure
(θd) is given by

$$\theta_d = 180^\circ + \arg[G(s)H(s)]$$

where arg G(s)H(s) is the angle of G(s)H(s) excluding the pole where the angle is to be calculated.

Similarly, the angle of arrival is given by

$$\theta_a = 180^\circ - \arg[G(s)H(s)]$$

where arg G(s)H(s) is the angle of G(s)H(s) excluding the zero where the angle is to be calculated.

Example 11.18 Find the angle of departure for G(s)H(s) = K(s + 3)/[s+ 2 +j) (s + 2 -j)] where K > 0.

Solution

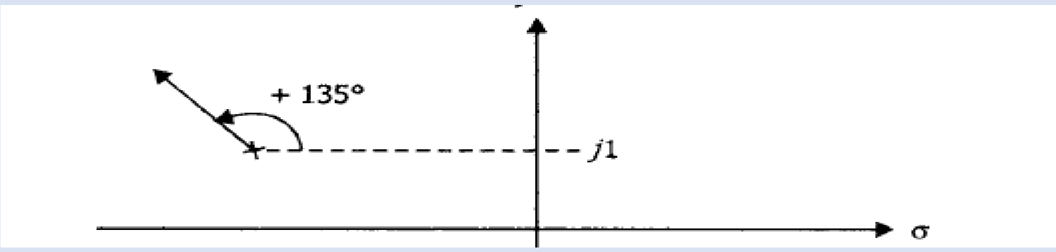
(i) Analytically:

To calculate (θd) analytically for s = -2 + j, the term s + 2 - j should be included from G(s)H(s).

$$G(s)H(s)\Big|_{s=-2+j} = \frac{K(-2+j+3)}{(-2+j+2+j)} = \frac{K(1+j)}{2j} = \frac{K\left(\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}\right)}{\sqrt{2}\angle 90^\circ}$$

$$= \frac{K\angle 45^\circ}{\sqrt{2}\angle 90^\circ} = \frac{K}{\sqrt{2}}\angle -45^\circ$$

$$\theta_d = 180^\circ + \arg[G(s)H(s)] = 180^\circ - 45^\circ = 135^\circ$$



8.5.4 Plotting and Calibrating the Root Locus

Once we sketch the root locus using the rules from Section 8.4, we may want to accurately locate points on the root locus as well as find their associated gain. For example, we might want to know the exact coordinates of the root locus as it crosses the radial line representing 20% overshoot. Further, we also may want the value of gain at that point.

Consider the root locus shown in Figure 8.12. Let us assume we want to find the exact point at which the locus crosses the 0.45 damping ratio line and the gain at that point. Figure 8.18 shows the system's open loop poles and zeros along with the $\zeta = 0.45$ line. If a few test points along the $\zeta = 0.45$ line are selected, we can evaluate their angular sum and locate that point where the angles add up to an odd multiple of 180°. It is at this point that the root locus exists. Equation (8.21) can then be used to evaluate the gain, K, at that point.

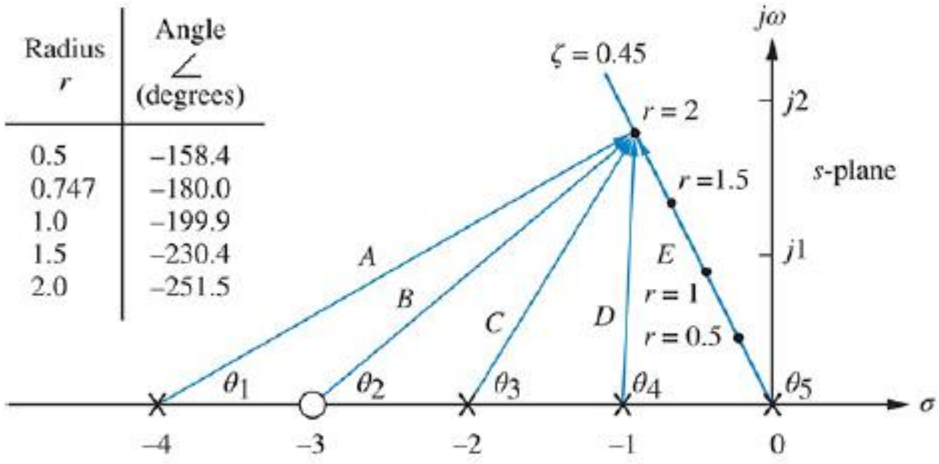


FIGURE 8.18 Finding and calibrating exact points on the root locus of Figure 8.12

Selecting the point at radius 2 ($r = 2$) on the $\zeta = 0.45$ line, we add the angles of the zeros and subtract the angles of the poles, obtaining

$$\theta 2 - \theta 1 - \theta 3 - \theta 4 - \theta 5 = -251.5° \quad 8.47$$

Since the sum is not equal to an odd multiple of 180°, the point at radius = 2 is not on the root locus. Proceeding similarly for the points at radius = 1.5, 1, 0.747, and 0.5, we obtain the table shown in Figure 8.18.

This table lists the points, giving their radius, r, and the sum of angles indicated by the symbol ∠. From the table, we see that the point at radius 0.747 is on the root locus, since the angles add up to −180°. Using Eq. (8.21), the gain, K, at this point is

$$K = \frac{|A|\,|C|\,|D|\,|E|}{|B|} = 1.71 \qquad (8.48)$$

8.6 Steps for Solving Problems on Root Locus

Step 1: Determine the branch number of loci, ending at infinity using Rule 1.

Step 2: Plot the poles and zeros on s-plane.

Step 3: Find real axis loci using Rule 2. Show the real axis loci wherever present by dark lines.

Step 4: Find the number of asymptotes and their angles by Rule 3.

Step 5: Using Rule 4, determine the centre of asymptotes and draw results of Steps 4 and 5.

Step 6: Determine the break-away/break-in point if present using Rule 6 and mark the point only.

Step 7: Determine jw crossover using Rule 7 if the locus crosses the jw axis.

Step 8: Calculate the angle of departure or the angle of arrival due to complex poles or zeros, respectively, using Rule 8.

Example 8.8 Sketch the root locus

$$G(s)H(s) = \frac{K}{s(s+1+j)(s+1-j)} \quad (K>0)$$

Solution

Step 1: Number of poles - n - 3 number of zeros = m = 0

number of loci - n - m = 3- 0 = 3

Step 2: The pole-zero plot of G(s)H(s) is shown in Fig. E l 1.13.

Step 3: Real axis locus is

(i) present for $-1 < \sigma < 0$

(ii) present for - ∞ < σ < - 1

This is shown in Fig. E l 1.13(a).

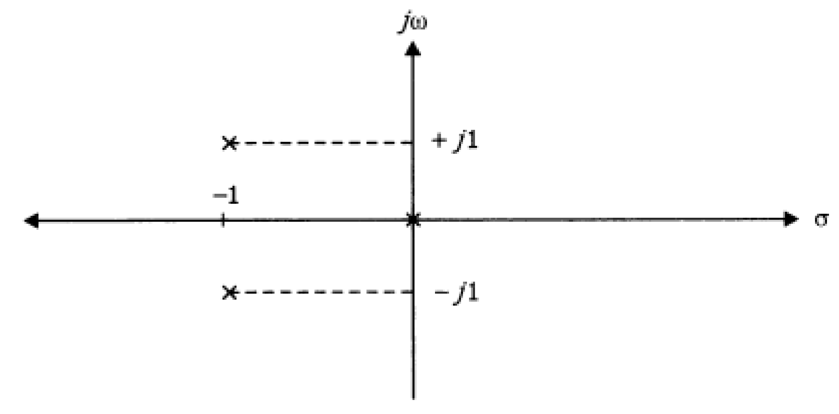


Fig. E11.13

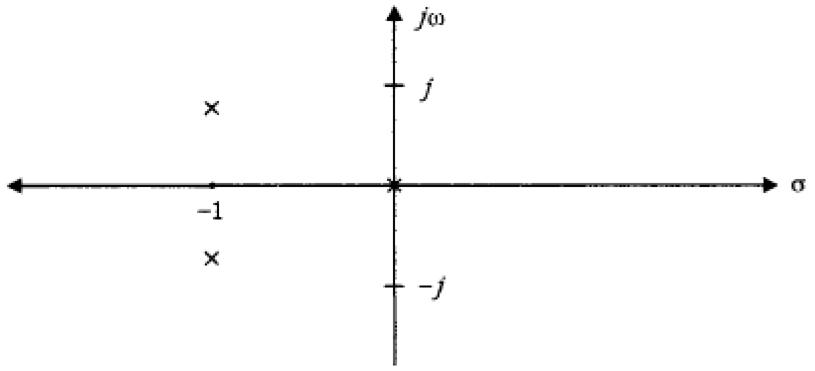


Fig. E11.13(a)

Step4: Since n-m = 3 and q = 0, 1,2,

$$\theta = \frac{(2\times0+1)\times180^\circ}{3} = 60^\circ,\ \theta_1 = \frac{(2\times1+1)\times180^\circ}{3} = 180^\circ \text{ and}$$

$$\theta_2 = \frac{(2\times2+1)\times180^\circ}{3} = 300^\circ$$

Step 5: Centroid (σ_c) $= \frac{(0-1-1)-0}{3} = \frac{-2}{3} = -0.667.$

Figure E11.13(b) shows the asymptotes.

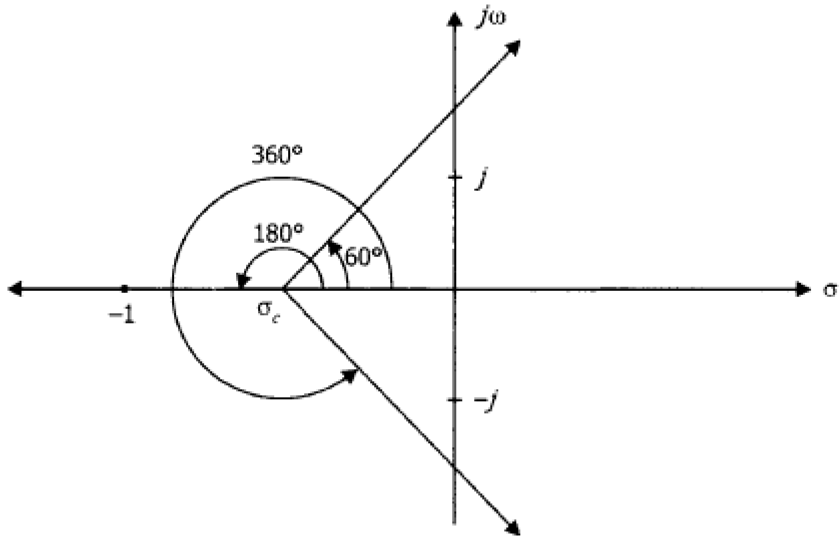


Fig. E11.13(b)

Step 6: Since the loci does not break-away from the real axis, calculation for break-away point is not required.

Step 7: $$1 + G(s)H(s) = 0$$

or $$1 + \frac{K}{s(s+1+j)(s+1-j)} = 0$$

or $$1 + \frac{K}{s(s^2+2s+2)} = 0$$

or $$s(s^2+2s+2) + K = 0$$

or $$s^3 + 2s^2 + 2s + K = 0$$

Routh's array

s^3	1	2
s^2	2	K
s^1	$\frac{4-K}{2}$	0
s^0	K	

To get K_{mar}

$$\frac{4-K_{mar}}{2} = 0$$

$\therefore$ $$A(s) = 2s^2 + K = 0$$

or $$2s^2 = -K = -4$$

$\therefore$ $$s^2 = -2$$

$\therefore$ $$s = \pm j\sqrt{2}$$

Step 8

$$G(s)H(s)\Big|_{s=-1+j} = \frac{K}{(-1+j)(-1+j+1+j)} = \frac{K}{(-1+j)(2j)} = \frac{K}{\sqrt{2}\left(\frac{-1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}\right)(j)}$$

$$= \frac{K}{\sqrt{2}\,\angle 135°\,\angle 90°} = \frac{K}{\sqrt{2}\angle +225°} = \frac{K}{\sqrt{2}}\angle -225°$$

$\therefore$ $G(s)H(s) = -225°$

$\therefore$ $\phi_0 = 180° + \arg\,[G(s)H(s)] = 180° - 225° = -45°.$

By symmetry, the angle of departure from $s = -1 - j$ is $+45°$.

Figure E11.13(c) shows the complete root locus for

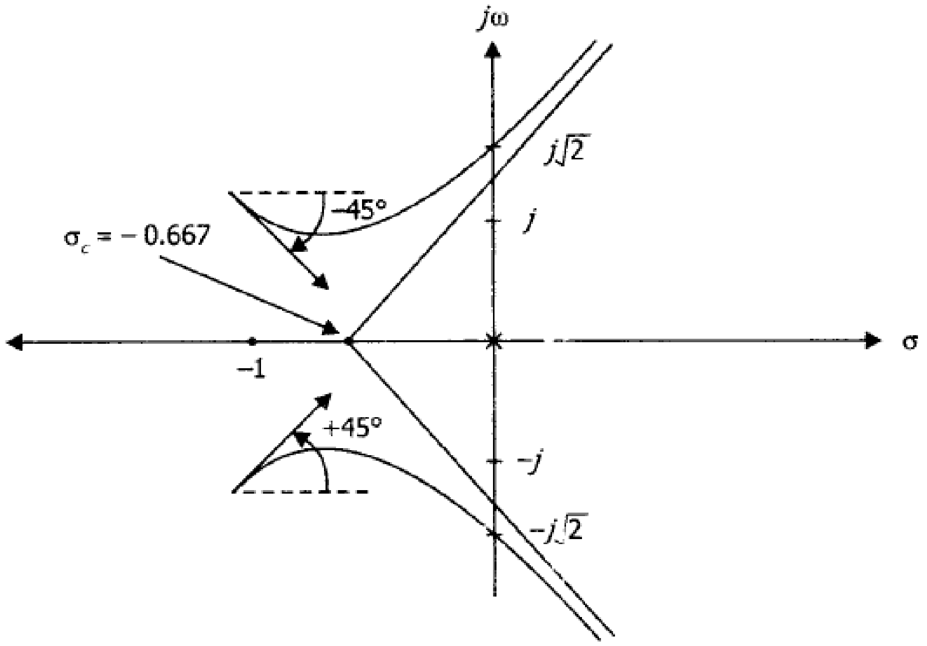


Fig. E11.13(c)