Tikrit university

Collage of Engineering Shirqat

Department of Electrical Engineering

Fourth Year

Control Engineering

# Chapter 2

# Lecture 6

# Transfer functions

Prepared by

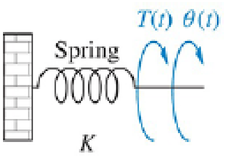
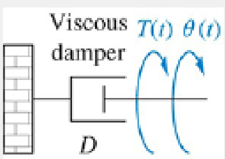
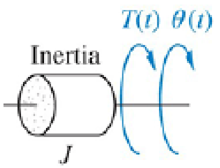
Asst Lecturer. Ahmed Saad Names

## 2.4 Rotational Mechanical System Transfer Functions

Having covered electrical and translational mechanical systems, we now move on to consider rotational mechanical systems. Rotational mechanical systems are handled the same way as translational mechanical systems, except that torque replaces force and angular displacement replaces translational displacement. The mechanical components for rotational systems are the same as those for translational systems, except that the components undergo rotation instead of translation. Table 2.5 shows the components along with the relationships between torque and angular velocity, as well as angular displacement. Notice that the symbols for the components look the same as translational symbols, but they are undergoing rotation and not translation.

**TABLE 2.5**

**Torque–angular velocity, torque–angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia**

| Component | Torque–angular velocity | Torque–angular displacement | Impedance $Z_M(s) = T(s)/\theta(s)$ |
|---|---|---|---|
|  | $T(t) = K\int_0^t \omega(\tau)\,d\tau$ | $T(t) = K\theta(t)$ | $K$ |
|  | $T(t) = D\omega(t)$ | $T(t) = D\frac{d\theta(t)}{dt}$ | $Ds$ |
|  | $T(t) = J\frac{d\omega(t)}{dt}$ | $T(t) = J\frac{d^2\theta(t)}{dt^2}$ | $Js^2$ |

Note: The following set of symbols and units is used throughout this book: T(t) − N-m (newton-meters), θ(t) − rad (radians), ω(t) − rad/s (radians/second), K − N-m/rad (newton-meters/radian), D − N-m-s/rad(newton-meters-seconds/radian). J −kg-m2 (kilograms-meters2 − newton-meters-seconds2/radian).

Also notice that the term associated with the mass is replaced by inertia. The values of K, D, and J are called spring constant, coefficient of viscous friction, and moment of inertia, respectively. The impedances of the mechanical components are also summarized in the last column of Table 2.5. The values can be found by taking the Laplace transform, assuming zero initial conditions, of the torque angular displacement column of Table 2.5.

The concept of degrees of freedom carries over to rotational systems, except that we test a point of motion by rotating it while holding still all other points of motion. The number of points of motion that can be rotated while all others are held still equals the number of equations of motion required to describe the system.

Writing the equations of motion for rotational systems is similar to writing them for translational systems; the only difference is that the free-body diagram consists of torques rather than forces. We obtain these torques using superposition. First, we rotate a body while holding all other points still and place on its free-body diagram all torques due to the body's own motion. Then, holding the body still, we rotate adjacent points of motion one at a time and add the torques due to the adjacent motion to the free-body diagram. The process is repeated for each point of motion. For each free-body diagram, these torques are summed and set equal to zero to form the equations of motion.

**Example 2.16 Transfer Function—Two Equations of Motion PROBLEM:**

Find the transfer function, θ2(s)/T(s), for the rotational system shown in Figure below. The rod is supported by bearings at either end and is undergoing torsion. A torque is applied at the left, and the displacement is measured at the right.

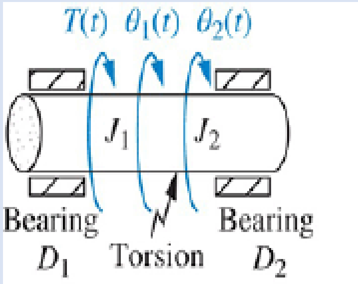



**Figure (a): Physical system**

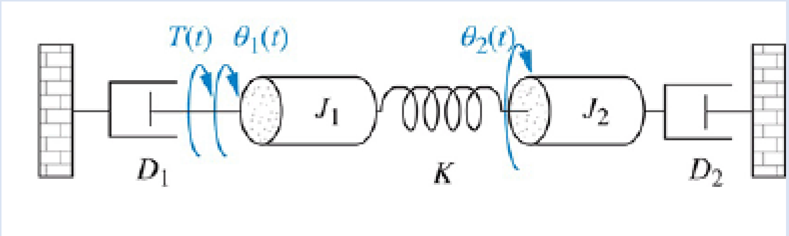



**figure (b) Schematic**

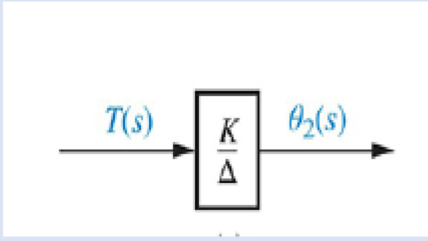



**figure (c) block diagram**

SOLUTION:

First, obtain the schematic from the physical system. Even though torsion occurs throughout the rod in Figure (a), we approximate the system by assuming that the torsion acts like a spring concentrated at one particular point in the rod, with an inertia J1 to the left and an inertia J2 to the right. We also assume that the damping inside the flexible shaft is negligible. The schematic is shown in Figure 2.22(b). There are two degrees of freedom, since each inertia can be rotated while the other is held still. Hence, it will take two simultaneous equations to solve the system. Next, draw a free-body diagram of J1 in below.

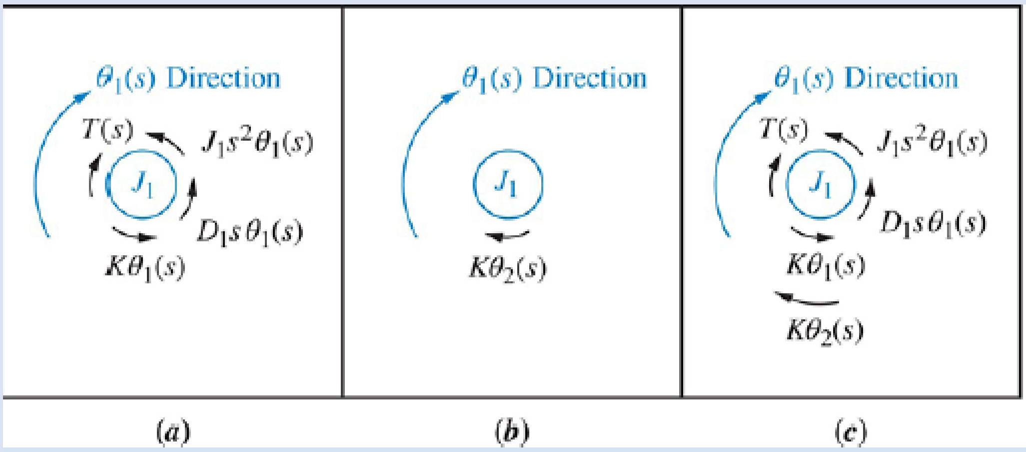



**FIGURE a. Torques on *J*1 due only to the motion of *J*1; b. torques on *J*1 due only to the motion of *J*2; c. final free-body diagram for *J*1**

using superposition. Figure (a) shows the torques on J1 if J2 is held still and J1 rotated. Figure (b) shows the torques on J1 if J1 is held still and J2 rotated. Finally, the sum of Figures 2.23(a) and 2.23(b) is shown in Figure 2.23(c), the final Freebody diagram for J1. The same process is repeated in Figure below for J2.

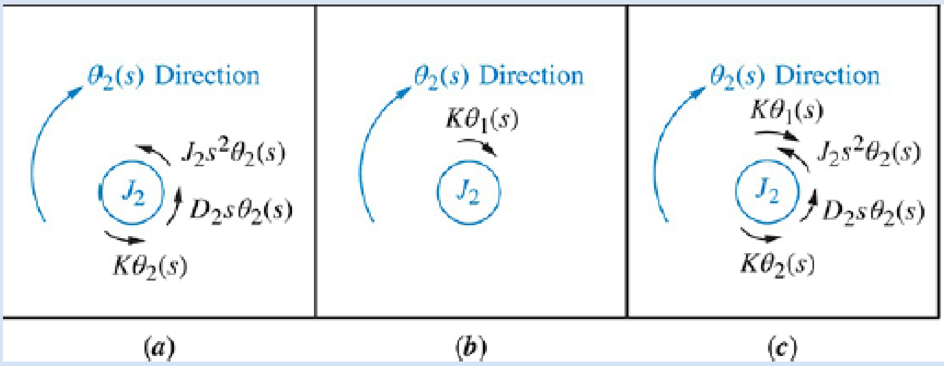



FIGURE **a. Torques on J2 due only to the motion of J2; b. torques on J2 due only to the motion of J1; c. final free-body diagram for J2**

Summing torques, respectively, from tow (c) Figures above we obtain the equations of motion,

$(J_1s^2 + D_1s + K)\,\theta_1(s) - K\theta_2(s) = T(s)$ **2.63**

$-K\theta_1(s) + (J_2s^2 + D_2s + K)\,\theta_2(s) = 0$ **2.64**

from which the required transfer function is found to be

$$\frac{\theta_2(s)}{T(s)} = \frac{K}{\Delta}$$ **2.65**

Where

$$\Delta = \begin{vmatrix} (J_1s^2 + D_1s + K) & -K \\ -K & (J_2s^2 + D_2s + K) \end{vmatrix}$$

## 2.5 Transfer Functions for Systems with Gears

Now that we are able to find the transfer function for rotational systems, we realize that these systems, especially those driven by motors, are rarely seen without associated gear trains driving the load. This section covers this important topic.

Gears provide mechanical advantage to rotational systems. Anyone who has ridden a 10-speed bicycle knows the effect of gearing. Going uphill, you shift to provide more torque and less speed. On the straightaway, you shift to obtain more speed and less torque. Thus, gears allow you to match the drive system and the load—a trade-off between speed and torque.

The linearized interaction between two gears is depicted in Figure below. An input gear with radius $r_1$ and $N_1$ teeth is rotated through angle $\theta_1(t)$ due to a torque, $T_1(t)$. An output gear with radius $r_2$ and $N_2$ teeth responds by rotating through angle $\theta_2(t)$ and delivering a torque, $T_2(t)$. Let us now find the relationship between the rotation of Gear 1, $\theta_1(t)$, and Gear 2, $\theta_2(t)$.

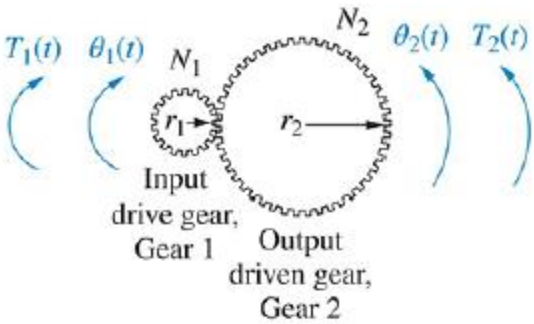



**FIGURE A gear system**

From Figure above, as the gears turn, the distance travelled along each gear's circumference is the same. Thus

$$r_1\theta_1 = r_2\theta_2 \qquad \textbf{2.66}$$

$$\frac{\theta 2}{\theta 1} = \frac{r1}{r2} = \frac{N1}{N2} \qquad \textbf{2.67}$$

What is the relationship between the input torque, $T_1$, and the delivered torque, $T_2$? If we assume the gears are *lossless*, that is, they do not absorb or store energy, the energy into Gear 1 equals the energy out of Gear Since the translational energy of force times displacement becomes the rotational energy of torque times angular displacement,

$$T_1\theta_1 = T_2\theta_2 \qquad \textbf{2.68}$$

$$\frac{T2}{T1} = \frac{\theta 1}{\theta 2} = \frac{N2}{N1} \qquad \textbf{2.69}$$

Thus, the torques are directly proportional to the ratio of the number of teeth. All results are summarized in Figure below.

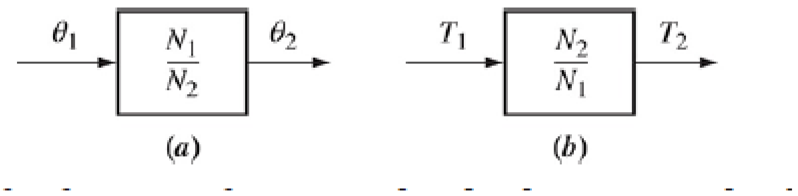
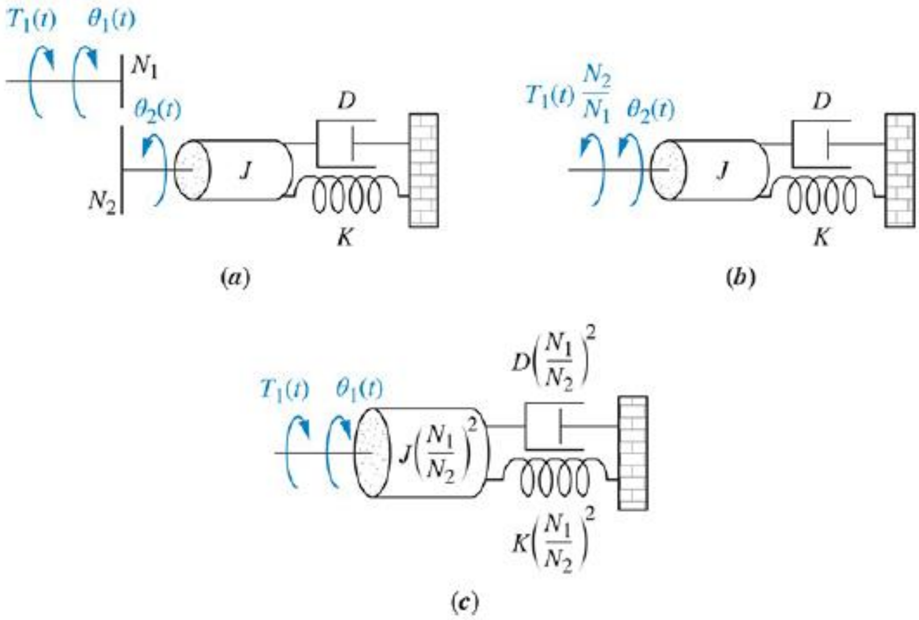



**FIGURE Transfer functions for a. angular displacement in lossless gears and b. torque in lossless gears**

Let us see what happens to mechanical impedances that are driven by gears. Figure below (a) shows gears driving a rotational inertia, spring, and viscous damper. For clarity, the gears are shown by an end-on view. We want to represent Figure below (a) as an equivalent system at $\theta_1$ without the gears. In other words, can the mechanical impedances be reflected from the output to the input, thereby eliminating the gears?




a. **Rotational system driven by gears; b. equivalent system at the output after reflection of input torque; c. equivalent system at the input after reflection of impedances**

From Figure of transfer function above (b), $T_1$ can be reflected to the output by multiplying by $N_2/N_1$. The result is shown in Figure above (b), from which we write the equation of motion as

$$\left(Js^2 + Ds + K\right)\ \theta_2(s) = T_1(s)\frac{N_2}{N_1} \qquad \textbf{2.70}$$

Now convert θ2(s) into an equivalent θ1(s), so that Eq. (2.70) will look as if it were written at the input. Using Figure of transfer function (a) to obtain θ2(s) in terms of θ1(s), we get

$$\left(Js^2 + Ds + K\right)\frac{N_1}{N_2}\theta_1(s) = T_1(s)\frac{N_2}{N_1} \qquad \textbf{2.71}$$

After simplification,

$$\left[J\left(\frac{N_1}{N_2}\right)^2 s^2 + D\left(\frac{N_1}{N_2}\right)^2 s + K\left(\frac{N_1}{N_2}\right)^2\right]\theta_1(s) = T_1(s) \qquad \textbf{2.72}$$

which suggests the equivalent system at the input and without gears shown in Figure (c). Thus, the load can be thought of as having been reflected from the output to the input. Generalizing the results, we can make the following statement: Rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedance by the ratio

$$\boxed{\left(\frac{\text{Number of teeth of gear on } \textit{destination} \text{ shaft}}{\text{Number of teeth of gear on } \textit{source} \text{ shaft}}\right)^2}$$

**Example 2.17 Transfer Function—System with Lossless Gears PROBLEM:**

Find the transfer function, θ2(s)/T1(s), for the system of Figure (a) below.

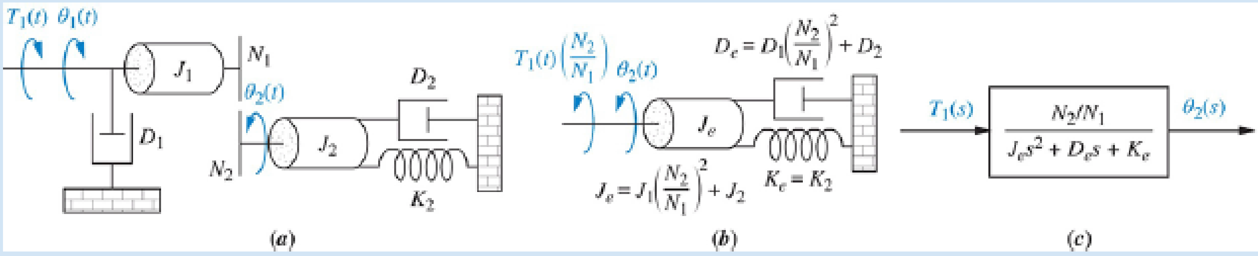



**FIGURE a. Rotational mechanical system with gears; b. system after reflection of torques and impedances to the output shaft; c. block diagram**

**SOLUTION:**

It may be tempting at this point to search for two simultaneous equations corresponding to each inertia. The inertias, however, do not undergo linearly independent motion, since they are tied together by the gears. Thus, there is only one degree of freedom and hence one equation of motion. Let us first reflect the impedances (*J*1 and *D*1) and torque (*T*1) on the input shaft to the output as shown in Figure (*b*), where the impedances are reflected by $(N2/N1)^2$ and the torque is reflected by (*N*2/*N*1). The equation of motion can now be written as

$$(J_e s^2 + D_e s + K_e)\,\theta_2(s) = T_1(s)\,\frac{N2}{N1} \qquad \textbf{2.73}$$

Where

$$J_e = J_1\left(\frac{N_2}{N_1}\right)^2 + J_2;\;\; D_e = D_1\left(\frac{N_2}{N_1}\right)^2 + D_2;\;\; K_e = K_2$$

Solving for θ2(s)/T1(s), the transfer function is found to be

$$G(s) = \frac{\theta_2(s)}{T_1(s)} = \frac{N_2/N_1}{J_e s^2 + D_e s + K_e} \qquad \textbf{2.74}$$

## 2.6 Electromechanical System Transfer Functions

In the last section we talked about rotational systems with gears, which completed our discussion of purely mechanical systems. Now, we move to systems that are hybrids of electrical and mechanical variables, the electromechanical systems. We have seen one application of an electromechanical system in Chapter 1, the antenna azimuth position control system. Other applications for systems with electromechanical components are robot controls, sun and star trackers, and computer tape and disk drive position controls.

A motor is an electromechanical component that yields a displacement output for a voltage input, that is, a mechanical output generated by an electrical input. We will derive the transfer function for one particular kind of electromechanical system, the armature-controlled dc servomotor. The motor's schematic is shown in Figure below (*a*), and the transfer function we will derive appears in Figure (*b*).

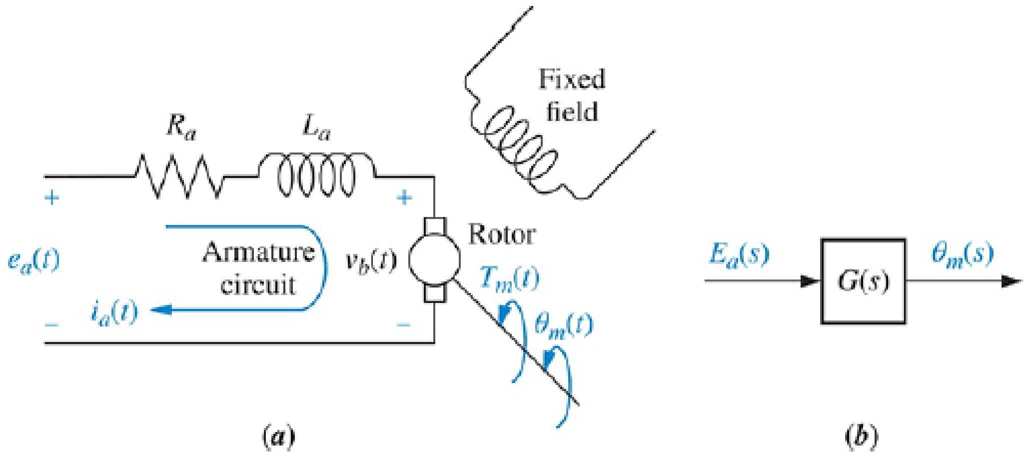



**FIGURE DC motor: a. schematic b. block diagram**

$$v_b(t) = K_b \frac{d\theta_m(t)}{dt} \quad \textbf{2.75}$$

We call vb(t) the back electromotive force(back emf); Kb is a constant of proportionality called the back emf constant; and $d\theta_m(t)/dt = \omega_m(t)$ is the angular velocity of the motor. Taking the Laplace transform, we get

$$V_b(s) = K_b s \theta_m(s) \quad \textbf{2.76}$$

The relationship between the armature current, $i_a(t)$, the applied armature voltage, $e_a(t)$, and the back emf, vb(t), is found by writing a loop equation around the Laplace transformed armature circuit.

$$R_a I_a(s) + L_a s I_a(s) + Vb(s) = E_a(s) \qquad \textbf{2.77}$$

The torque developed by the motor is proportional to the armature current; thus,

$$Tm(s) = KtIa(s) \qquad \textbf{2.78}$$

where Tm is the torque developed by the motor, and Kt is a constant of proportionality, called the motor torque constant, which depends on the motor and magnetic field characteristics. In a consistent set of units, the value of Kt is equal to the value of Kb. Rearranging Eq. (2.78) yields

$$I_a(s) = \frac{1}{K_t} T_m(s) \qquad \textbf{2.79}$$

To find the transfer function of the motor, we first substitute Eqs. (2.76) and (2.79) into (2.77), yielding

$$\frac{(R_a + L_a s) T_m(s)}{K_t} + K_b s \theta_m(s) = E_a(s) \qquad \textbf{2.80}$$

Now we must find Tm(s) in terms of θm(s) if we are to separate the input and output variables and obtain the transfer function, θm(s)/Ea(s).

Figure below shows a typical equivalent mechanical loading on a motor. Jm is the equivalent inertia at the armature and includes both the armature inertia and, as we will see later, the load inertia reflected to the armature. Dm is the equivalent viscous damping at the armature and includes both the armature viscous damping and, as we will see later, the load viscous damping reflected to the armature. From Figure below,

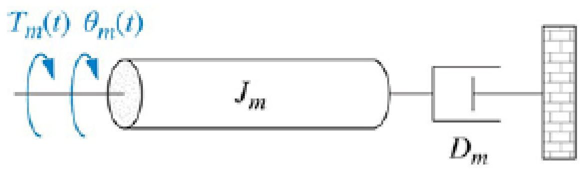



**FIGURE Typical equivalent mechanical loading on a motor**

$$Tm(s) = (Jms^2 + Dms)\,\theta m(s) \quad 2.81$$

Substituting Eq. (2.81) into Eq. (2.80) yields

$$\frac{(R_a + L_a s)\left(J_m s^2 + D_m s\right)\theta_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s) \quad 2.82$$

If we assume that the armature inductance, La, is small compared to the armature resistance, Ra, which is usual for a dc motor, Eq. (2.82) becomes

$$\left[\frac{R_a}{K_t}(J_m s + D_m) + K_b\right] s\theta_m(s) = E_a(s) \quad 2.83$$

After simplification, the desired transfer function, $\theta m(s)/Ea(s)$, is found to be

$$\boxed{\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]}} \quad 2.84$$

Even though the form of Eq. (2.84) is relatively simple, namely

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K}{s(s+\alpha)} \quad 2.85$$

the reader may be concerned about how to evaluate the constants.

Let us first discuss the mechanical constants, *Jm* and *Dm*. Consider Figure below, which shows a motor with inertia *Ja* and damping *Da* at the armature driving a load consisting of inertia *JL* and damping *DL*. Assuming that all inertia and damping values shown are known, *JL* and *DL* can be reflected back to the armature as some equivalent inertia and damping to be added to *Ja* and *Da*, respectively. Thus, the equivalent inertia, *Jm*, and equivalent damping, *Dm*, at the armature are

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2; \quad D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2 \quad 2.86$$

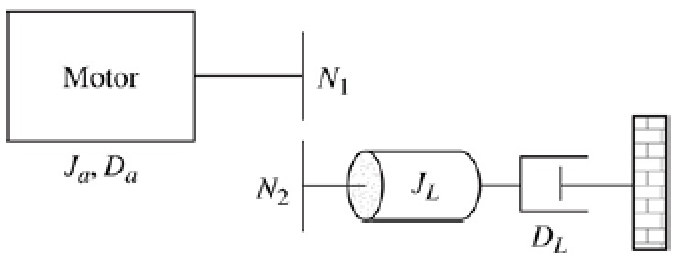



**FIGURE DC motor driving a rotational mechanical load**

Now that we have evaluated the mechanical constants, *Jm* and *Dm*, what about the electrical constants in the transfer function of Eq. (2.84)? We will show that these constants can be obtained through a dynamometer test of the motor, where a dynamometer measures the torque and speed of a motor under the condition of a constant applied voltage. Let us first develop the relationships that dictate the use of a dynamometer. Substituting Eqs. (2.76) and (2.79) into Eq. (2.77), with *La* = 0, yields

$$\frac{R_a}{K_t}T_m(s) + K_b s\theta_m(s) = E_a(s) \qquad 2.87$$

Taking the inverse Laplace transform, we get

$$\frac{R_a}{K_t}T_m(t) + K_b\omega_m(t) = e_a(t) \qquad 2.88$$

where the inverse Laplace transform of $s\theta_m(s)$ is $d\theta_m(t)/dt$ or, alternately, $\omega_m(t)$. If a dc voltage, $e_a$, is applied, the motor will turn at a constant angular velocity, $\omega_m$, with a constant torque, Tm. Hence, dropping the functional relationship based on time from Eq. (2.88), the following relationship exists when the motor is operating at steady state with a dc voltage input:

$$\frac{R_a}{K_t}T_m + K_b\omega_m = e_a \qquad 2.89$$

Solving for *Tm* yields

$$\boxed{T_m = -\frac{K_bK_t}{R_a}\omega_m + \frac{K_t}{R_a}e_a} \qquad 2.90$$

Equation (2.90) is a straight line, $T_m$ vs. $\omega_m$, and is shown in Figure below. This plot is called the torque–speed curve. The torque axis intercept occurs when the angular velocity reaches zero. That value of torque is called the stall torque, Tstall. Thus,

$$T_{\text{stall}} = \frac{K_t}{R_a}e_a \qquad 2.91$$

The angular velocity occurring when the torque is zero is called the *no-load speed*, $\omega_{\text{no-load}}$. Thus,

$$\omega_{\text{no-load}} = \frac{e_a}{K_b} \qquad 2.92$$

The electrical constants of the motor's transfer function can now be found from Eqs. (2.91) and (2.92) as

$$\boxed{\frac{K_t}{R_a} = \frac{T_{\text{stall}}}{e_a}} \qquad \textbf{2.93}$$

and

$$\boxed{K_b = \frac{e_a}{\omega_{\text{no-load}}}} \qquad \textbf{2.94}$$

The electrical constants, $K_t/R_a$ and $K_b$, can be found from a dynamometer test of the motor, which would yield Tstall and $\omega_{\text{no-load}}$ for a given ea.

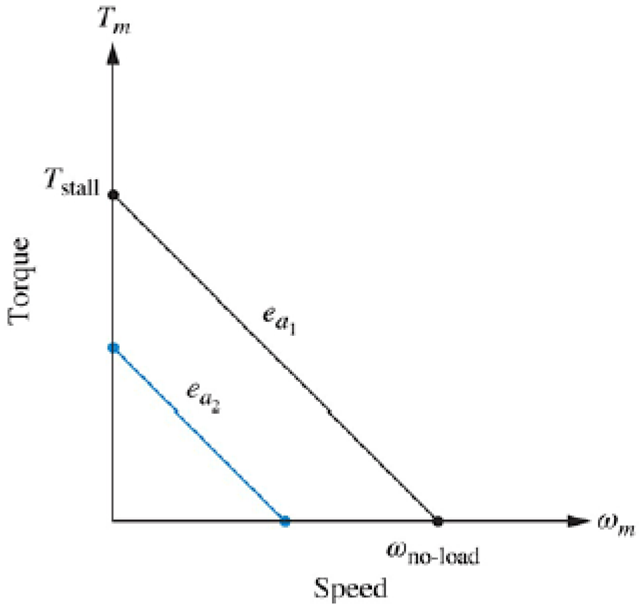



**FIGURE Torque–speed curves with an armature voltage, $e_a$, as a parameter**

Example 2.23 Transfer Function—DC Motor and Load

PROBLEM:

Given the system and torque–speed curve of Figure below(a) and (b), find the transfer function,

$\theta L(s)/Ea(s)$.

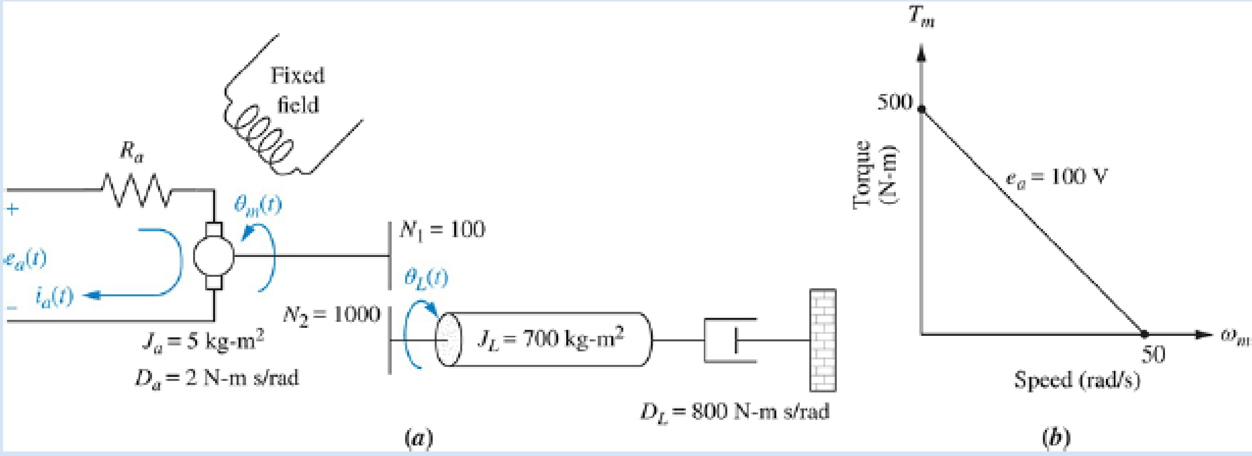



**FIGURE a. DC motor and load; b. torque–speed curve; c. block diagram**

SOLUTION:

Begin by finding the mechanical constants, $J_m$ and $D_m$, in Eq. (2.84). From Eq. (2.86), the total inertia at the armature of the motor is

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2 = 5 + 700\left(\frac{1}{10}\right)^2 = 12 \qquad \textbf{2.95}$$

and the total damping at the armature of the motor is

$$D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2 = 2 + 800\left(\frac{1}{10}\right)^2 = 10 \qquad \textbf{2.96}$$

Now we will find the electrical constants, Kt/Ra and Kb. From the torque–speed curve of Figure (b),

Tstall = 500 $\omega_{no-load} = 50$ $e_a = 100$ Hence the electrical constants are

$$\frac{K_t}{R_a} = \frac{T_{\text{stall}}}{e_a} = \frac{500}{100} = 5 \qquad \textbf{2.97}$$

$$K_b = \frac{e_a}{\omega_{\text{no-load}}} = \frac{100}{50} = 2 \qquad \textbf{2.98}$$

Substituting Eqs. (2.95), (2.96), (2.97), and (2.98) into Eq. (2.84) yield

$$\frac{\theta_m(s)}{E_a(s)} = \frac{5/12}{s\left\{s + \frac{1}{12}[10 + (5)(2)]\right\}} = \frac{0.417}{s(s+1.667)} \qquad \textbf{2.99}$$

In order to find $\theta L(s)/Ea(s)$, we use the gear ratio, $N1/N2 = 1/10$, and find

$$\frac{\theta_L(s)}{E_a(s)} = \frac{0.0417}{s(s+1.667)} \qquad \textbf{2.100}$$