Tikrit university

Collage of Engineering Shirqat

Department of Electrical Engineering

Fourth Year

Control Engineering

Chapter 2

Lecture 5

Transfer functions

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Operational Amplifiers

An operational amplifier, pictured in Figure below (a), is an electronic amplifier used as a basic building block to implement transfer functions. It has the following characteristics:

1. Differential input, $v_2(t) - v_1(t)$
2. High input impedance, $Zi = \infty$ (ideal)
3. Low output impedance, $Z_o = 0$ (ideal)
4. High constant gain amplification, $A = \infty$ (ideal)

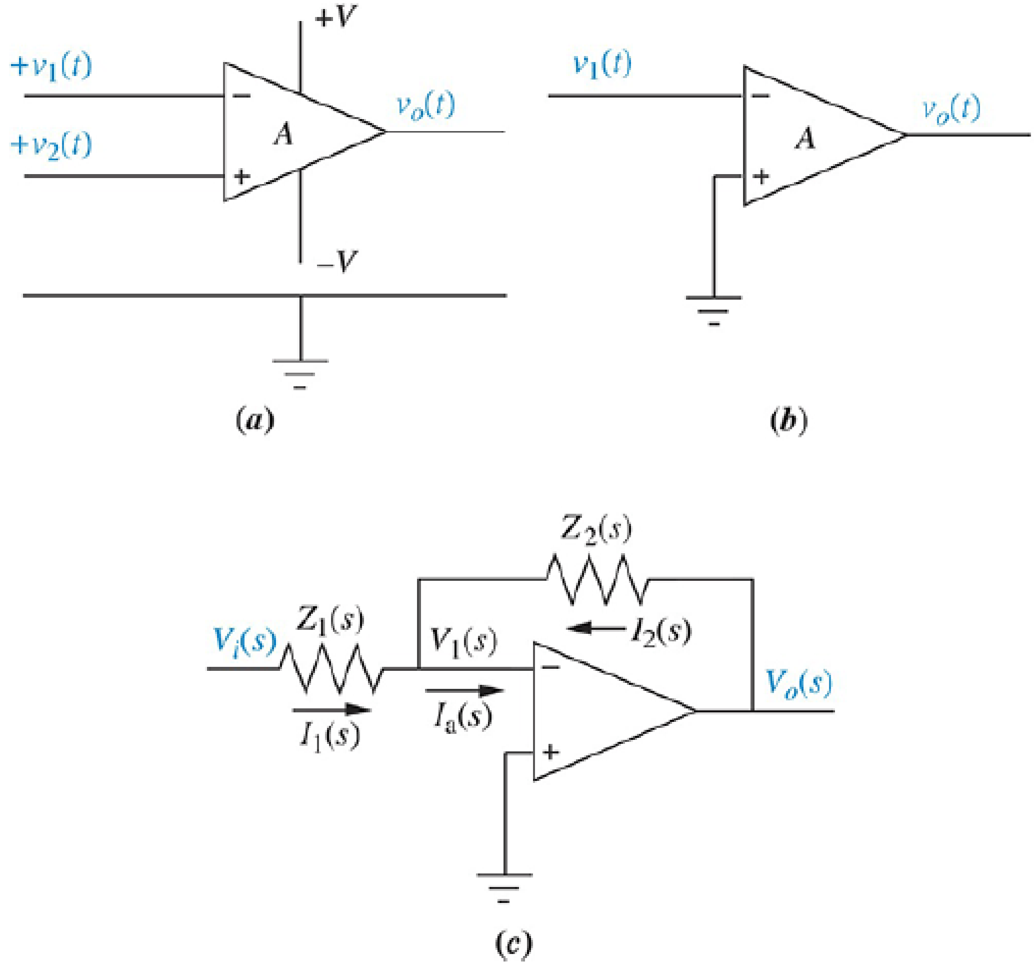


FIGURE a. Operational amplifier; b. schematic for an inverting operational amplifier; c. inverting operational amplifier configured for transfer function realization.

Typically, the amplifier gain, A, is omitted. The output, vo(t), is given by

$$v_o(t) = A(v_2(t) - v_1(t)) \quad \textbf{2.38}$$

Inverting Operational Amplifier

If $v_2(t)$ is grounded, the amplifier is called an *inverting operational amplifier*, as shown in Figure (*b*). For the inverting operational amplifier, we have

$$V_o(t) = -Av_1(t) \qquad 2.39$$

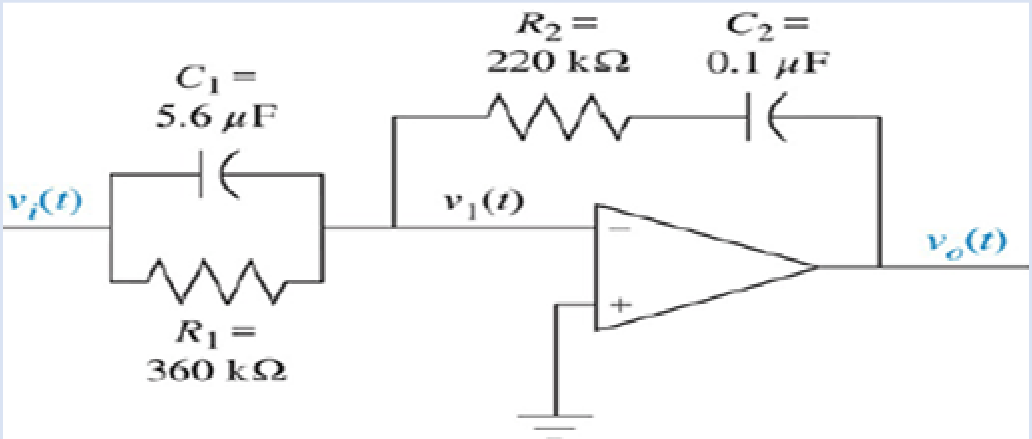
If two impedances are connected to the inverting operational amplifier as shown in Figure (*c*), we can derive an interesting result if the amplifier has the characteristics mentioned in the beginning of this subsection. If the input impedance to the amplifier is high, then by Kirchhoff's current law $Ia(s) = 0$ and $I_1(s) = -I_2(s)$. Also, since the gain A is large, $v_1(t) \approx 0$. Thus, $I_1(s) = V_i(s)/Z_1(s)$, and $-I_2(s) = -Vo(s)/Z_2(s)$. Equating the two currents, $Vo(s)/Z_2(s) = -V_i(s)/Z_1(s)$, or the transfer function of the inverting operational amplifier configured as shown in Figure (*c*) is

$$\boxed{\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}} \qquad 2.40$$

Example 2.12 Transfer Function—Inverting Operational Amplifier Circuit

PROBLEM:

Find the transfer function, $Vo(s)/Vi(s)$, for the circuit given in Figure below



SOLUTION:

The transfer function of the operational amplifier circuit is given by Eq. (2.40). Since the admittances of parallel components add, *Z*1(*s*) is the reciprocal of the sum of the admittances, or

$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}} = \frac{1}{5.6 + 10^{-6}s + \frac{1}{360\times10^3}} = \frac{360\times10^3}{2.016s+1} \qquad 2.41$$

For Z2(s) the impedances add, or

$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220\times10^3 + \frac{10^7}{s} \qquad 2.42$$

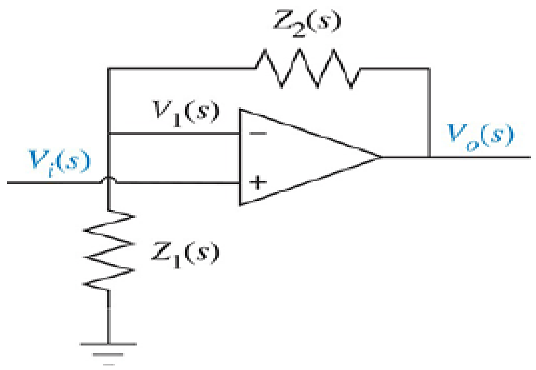
Substituting Eqs. (2.41) and (2.42) into Eq. (2.40) and simplifying, we get

$$\frac{V_o(s)}{V_i(s)} = -1.232\frac{s^2+45.95s+22.55}{s} \qquad 2.43$$

The resulting circuit is called a PID controller and can be used to improve the performance of a control system.

Noninverting Operational Amplifier

Another circuit that can be analysed for its transfer function is the noninverting operational amplifier circuit shown in Figure below.



We now derive the transfer function. We see that

$V_o(s) = A(V_i(s) - V_1(s))$ **2.44**

But, using voltage division,

$$V_1(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)} V_o(s)$$ **2.45**

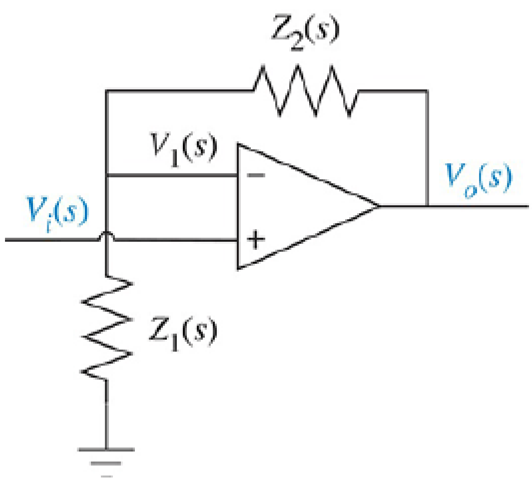
Substituting Eq. (2.45) into Eq. (2.44), rearranging, and simplifying, we obtain

$$\frac{V_o(s)}{V_i(s)} = \frac{A}{1 + AZ_1(s)/(Z_1(s) + Z_2(s))}$$ **2.46**

For large *A*, we disregard unity in the denominator and Eq. (2.46) becomes

$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}}$$ **2.47**

Let us now look at an example.



General noninverting operational amplifier circuit

Example 2.13 Transfer Function—Noninverting Operational Amplifier Circuit

PROBLEM:

Find the transfer function, *Vo*(*s*)/*Vi*(*s*), for the circuit given in Figure below.

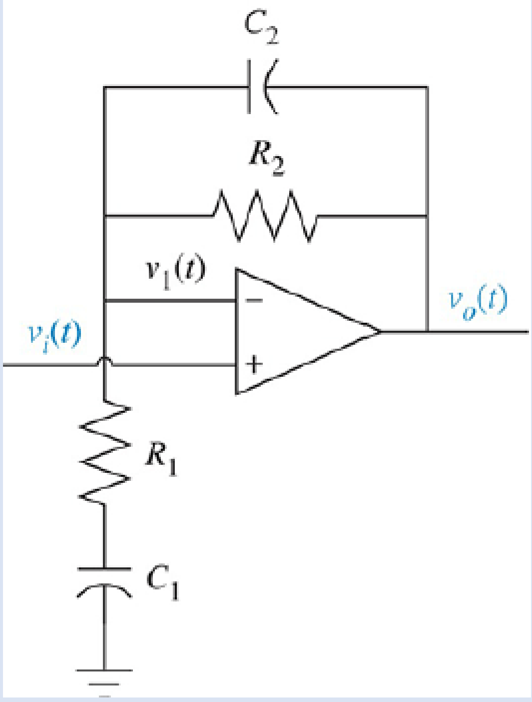


FIGURE Noninverting operational amplifier circuit for Example 2.13

SOLUTION:

We find each of the impedance functions, Z1(s) and Z2(s), and then substitute them into Eq.(2.47). Thus,

$$Z_1(s) = R_1 + \frac{1}{C_1 s} \qquad \textbf{2.48}$$

$$Z_2(s) = \frac{R_2(1/C_2 s)}{R_2 + (1/C_2 s)} \qquad \textbf{2.49}$$

Substituting Eqs. (2.48) and (2.49) into Eq. (2.47) yields

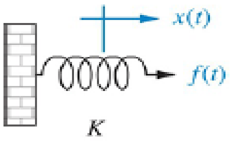
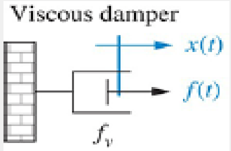
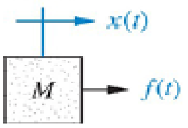
$$\frac{V_o(s)}{V_i(s)} = \frac{C_2C_1R_2R_1s^2 + (C_2R_2 + C_1R_2 + C_1R_1)s + 1}{C_2C_1R_2R_1s^2 + (C_2R_2 + C_1R_1)s + 1} \qquad \textbf{2.50}$$

2.3 Translational Mechanical System Transfer Functions

Mechanical systems, like electrical networks, have three passive, linear components. Two of them, the spring and the mass, are energy-storage elements; one of them, the viscous damper, dissipates energy. The two energy-storage elements are analogous to the two electrical energy-storage elements, the inductor and capacitor. The energy dissipator is analogous to electrical resistance. Let us take a look at these mechanical elements, which are shown in Table 2.4. In the table, K, fv, and M are called spring constant, coefficient of viscous friction, and mass, respectively.

TABLE 2.4

Force–velocity, force–displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force–velocity	Force–displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: f(t) = N (newtons), x(t) = m (meters), v(t) = m/s (meters/second), K = N/m (newtons/meter), fv = N-s/m (newton-seconds/meter), M = kg (kilograms = newtonseconds^2/ meter).

We now create analogies between electrical and mechanical systems by comparing Tables 2.3 and 2.4. Comparing the force–velocity column of Table 2.4 to the voltage–current column of Table 2.3, we see that mechanical force is analogous to electrical voltage and mechanical velocity is analogous to electrical current. Comparing the force–displacement column of Table 2.4 with the voltage–charge column of Table 2.3 leads to the analogy between the mechanical displacement and electrical charge. We also see that the spring is analogous to the capacitor, the viscous damper is analogous to the resistor, and the mass is analogous to the inductor. Thus, summing forces written in terms of velocity is analogous to summing voltages written in terms of current, and the resulting mechanical differential equations are analogous to mesh equations. If the forces are written in terms of displacement, the resulting mechanical equations resemble, but are not analogous to, the mesh equations. We, however, will use this model for mechanical systems so that we can write equations directly in terms of displacement.

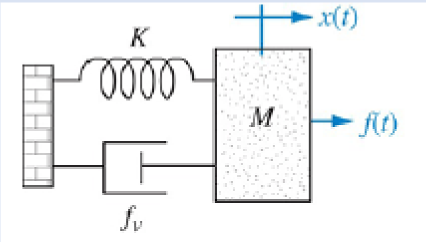
Another analogy can be drawn by comparing the force–velocity column of Table 2.4 to the current–voltage column of Table 2.3 in reverse order. Here the analogy is between force and current and between velocity and voltage. Also, the spring is analogous to the inductor, the viscous damper is analogous to the resistor, and the mass is analogous to the capacitor. Thus, summing forces written in terms of velocity is analogous to summing currents written in terms of voltage and the resulting mechanical differential equations are analogous to nodal equations.

The mechanical system requires just one differential equation, called the equation of motion, to describe it. We will begin by assuming a positive direction of motion, for example, to the right. This assumed positive direction of motion is similar to assuming a current direction in an electrical loop. Using our assumed direction of positive motion, we first draw a free-body diagram, placing on the body all forces that act on the body either in the direction of motion or opposite to it. Next, we use Newton's law to form a differential equation of motion by summing the forces and setting the sum equal to zero. Finally, assuming zero initial conditions, we take the Laplace transform of the differential equation, separate the variables, and arrive at the transfer function. An example follows.

Example 2.14 Transfer Function—One Equation of Motion

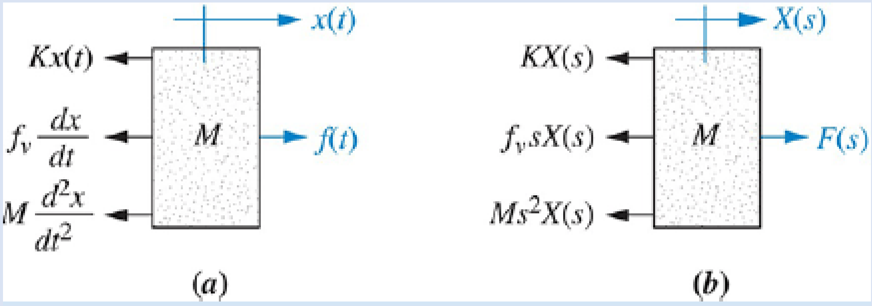
PROBLEM:

Find the transfer function, $X(s)/F(s)$, for the system of Figure below.



SOLUTION:

Begin the solution by drawing the free-body diagram shown in Figure below(*a*). Place on the mass all forces felt by the mass. We assume the mass is traveling toward the right. Thus, only the applied force points to the right; all other forces impede the motion and act to oppose it. Hence, the spring, viscous damper, and the force due to acceleration point to the left



a. Free-body diagram of mass, spring, and damper system; b. transformed free-body diagram

We now write the differential equation of motion using Newton's law to sum to zero all of the forces shown on the mass in Figure (a):

$$M\frac{d^2x(t)}{dt^2} + f_v\frac{dx(t)}{dt} + Kx(t) = f(t) \quad 2.51$$

Taking the Laplace transform, assuming zero initial conditions,

$$Ms^2X(s) + f_v sX(s) + KX(s) = F(s) \quad 2.52$$

or

$$(Ms^2 + f_v s + K)X(s) = F(s) \quad 2.53$$

Solving for the transfer function yields

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K} \quad 2.54$$

Now can we parallel our work with electrical networks by circumventing the writing of differential equations and by defining impedances for mechanical components? If so, we can apply to mechanical systems the problem-solving techniques learned in the previous section. Taking the Laplace transform of the force–displacement column in Table 2.4, we obtain for the spring,

$$\boxed{F(s) = KX(s)} \quad \textbf{2.55}$$

for the viscous damper,

$$\boxed{F(s) = f_v sX(s)} \quad \textbf{2.56}$$

and for the mass,

$$\boxed{F(s) = Ms^2X(s)} \quad \textbf{2.57}$$

If we define impedance for mechanical components as

$$\boxed{Z_M(s) = \frac{F(s)}{X(s)}} \quad \textbf{2.58}$$

$$\boxed{F(s) = Z_M(s)X(s)} \quad \textbf{2.59}$$

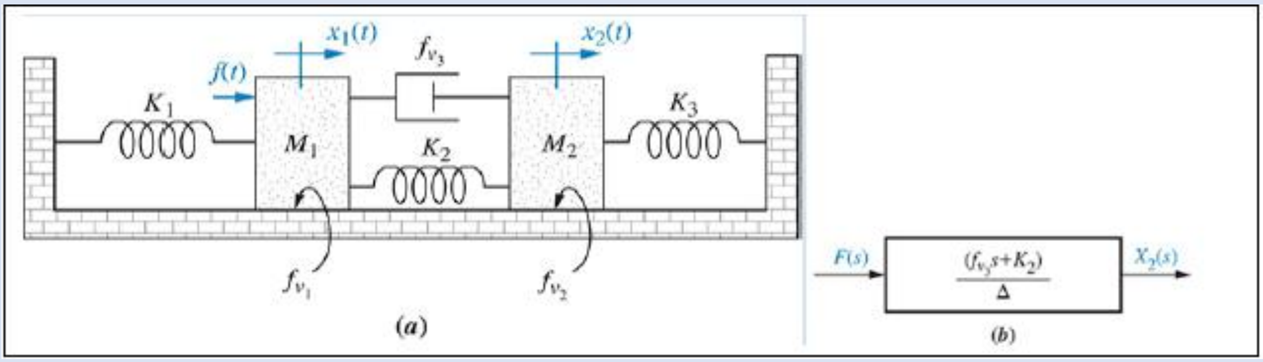
$$[\text{Sum of impedances}]\, X(s) = [\text{Sum of applied forces}] \quad \textbf{2.60}$$

Many mechanical systems are similar to multiple-loop and multiple-node electrical networks, where more than one simultaneous differential equation is required to describe the system. In mechanical systems, the number of equations of motion required is equal to the number of linearly independent motions. Linear independence implies that a point of motion in a system can still move if all other points of motion are held still.

In order to work such a problem, we draw the free-body diagram for each point of motion and then use superposition. For each free-body diagram we begin by holding all other points of motion still and finding the forces acting on the body due only to its own motion. Then we hold the body still and activate the other points of motion one at a time, placing on the original body the forces created by the adjacent motion. Using Newton's law, we sum the forces on each body and set the sum to zero. The result is a system of simultaneous equations of motion. As Laplace transforms, these equations are then solved for the output variable of interest in terms of the input variable from which the transfer function is evaluated. Example 2.15 demonstrates this problem-solving technique.

Example 2.15 Transfer Function—Two Degrees of Freedom

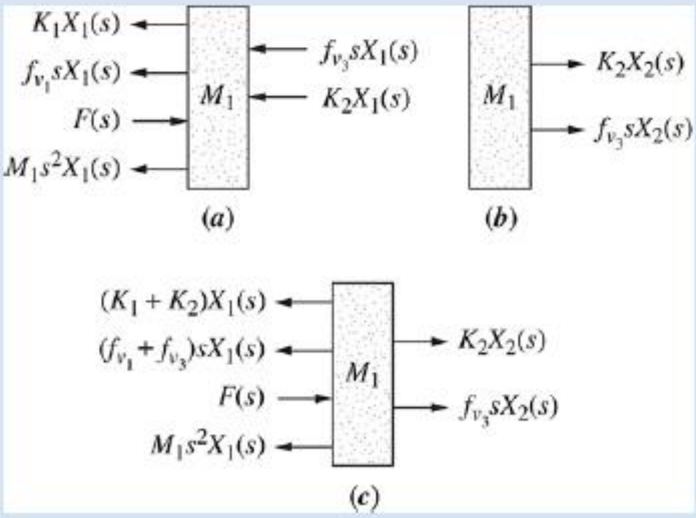
Find the transfer function, $X2(s)/F(s)$, for the system of Figure below (*a*).



a. Two-degrees-of-freedom translational mechanical system **b. block diagram**

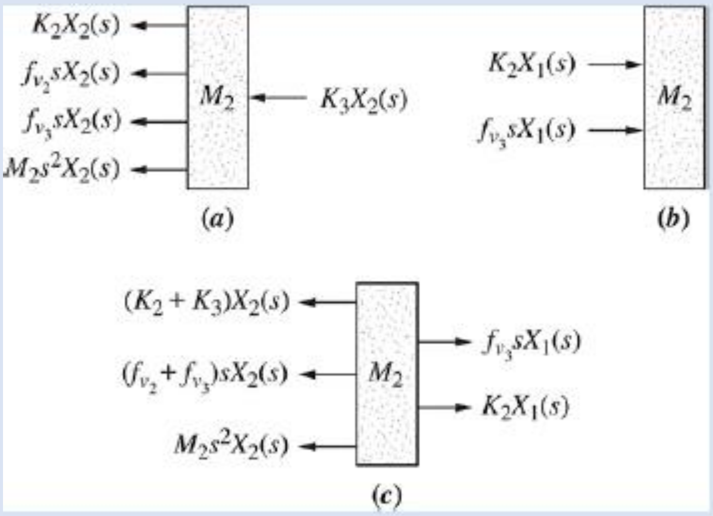
SOLUTION:

The system has two degrees of freedom, since each mass can be moved in the horizontal direction while the other is held still. Thus, two simultaneous equations of motion will be required to describe the system. The two equations come from free-body diagrams of each mass. Superposition is used to draw the free-body diagrams. For example, the forces on *M*1 are due to (1) its own motion and (2) the motion of *M*2 transmitted to *M*1 through the system. We will consider these two sources separately. If we hold *M*2 still and move *M*1 to the right, we see the forces shown in Figure below (*a*). If we hold *M*1 still and move *M*2 to the right, we see the forces shown in Figure below (*b*). The total force on *M*1 is the superposition, or sum, of the forces just discussed. This result is shown in Figure below(*c*).



a. Forces on *M*1 due only to motion of *M*1; b. forces on *M*1 due only to motion of *M*2; c. all forces on *M*1

For *M*2, we proceed in a similar fashion: First we move *M*2 to the right while holding *M*1 still; then we move *M*1 to the right and hold *M*2 still. For each case we evaluate the forces on M2. The results appear in Figure below.



a. Forces on *M*2 due only to motion of *M*2; b. forces on *M*2 due only to motion of *M*1; c. all forces on *M*2

The Laplace transform of the equations of motion can now be written from Figures

$$\left[M_1 s^2 \left(F_{v_1} + f_{v_3}\right) s + \left(K_1 + K_2\right)\right] X_1(s) - \left(f_{v_3} s + K_2\right) X_2(s) = F(s) \quad \mathbf{2.60}$$

$$-\left(f_{v_3} s + K_2\right) X_1(s) + \left[M_2 s^2 + \left(f_{v_2} + f_{v_3}\right) s + \left(K_2 + K_3\right)\right] X_2(s) = 0 \quad \mathbf{2.61}$$

From this, the transfer function, *X*2(*s*)/*F*(*s*), is

$$\frac{X_2(s)}{F(s)} = G(s) = \frac{\left(f_{v_3} s + K_2\right)}{\Delta} \quad \mathbf{2.62}$$