Tikrit university

Collage of Engineering Shirqat

Department of Electrical Engineering

Fourth Class

Control Engineering

Chapter 5

Lecture 11

Time domain response

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5.6 ANALYSIS OF SECOND-ORDER SYSTEM

Figure 5.14 shows a general second-order system.

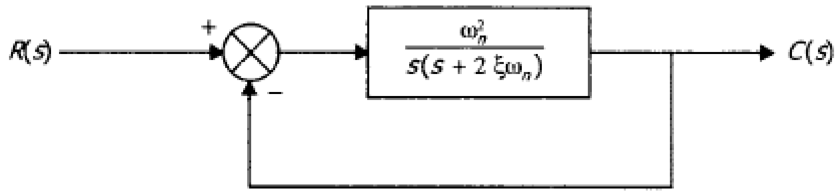

Fig. 5.14 Second-order system

From Fig. 5.14

$$G(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)}$$

$$G(s)H(s) = \frac{\omega_n^2}{s^2+2s\xi\omega_n} \qquad (5.24)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{\omega_n^2}{s^2+2s\xi\omega_n}}{1+\dfrac{\omega_n^2}{s^2+2s\xi\,\omega_n}} = \frac{\omega_n^2}{s^2+2s\xi\,\omega_n+\omega_n^2} \qquad (5.25)$$

Let us study the effect of pole location before finding the unit step response. To find the poles of closed loop transfer function of Fig. 5.14, let us put

$$s^2+2s\,\xi\omega_n+\omega_n^2 = 0$$

$$s = \frac{-2\xi\,\omega_n \pm \sqrt{(2\xi\,\omega_n)^2-4\omega_n^2}}{2} = -\,\xi\omega_n \pm \sqrt{\xi^2\omega_n^2-\omega_n^2}$$

$$s = -\,\xi\omega_n \pm \omega_n\sqrt{\xi^2-1} \qquad (5.26)$$

From Eq. (5.26), the following conclusions can be drawn:

(i) The poles are real and unequal if $\sqrt{\xi^2-1} > 0$
i.e., $\xi > 1$

(ii) The poles are real and equal if $\sqrt{\xi^2-1} = 0$
i.e., $\xi = 1$

(iii) The poles are real and complex conjugate if $\sqrt{\xi^2-1} < 0$
i.e., $\xi < 1$

where ξ is known as damping factor.

5. 7 Effect of ξ On Second-Order System

Let us analyse the effect of ξ, on pole location as well as response of second-order systems.

5.7.1 Effect of ξ on pole location

(i) Case 1: $0 < \xi < 1$

(a) The poles are complex conjugates of each other and are given by

$$s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{\xi^2 - 1}$$

(b) The poles are located on the second and third coordinates due to the existence of both real and imaginary parts.

(c) The response is underdamped.

(ii) Case 2: $\xi = 1$

(a) The poles are real and equal. The poles are $s_1 = s_2 = -\xi w_n$

(b) They lie on the negative a-axis.

(c) The response is critically damped

(iii) Case 3: $\xi > 1$

(a) The poles are real and unequal. These are given by

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

(b) Since there are no imaginary terms ($\xi > 1$), the poles lie on the negative a-axis and at unequal places.

(c) The response is overdamped.

(iv) Case 4: $\xi = 0$

(a) The poles are complex with only the imaginary part and lie on jw axis. The poles are conjugates of each other.

(b) The poles are given by $s_1 = s_2 = jw_n$

(c) The response is undamped.

(v) Case 5: **$0 > \xi > -1$**

(a) Since ξ, is negative, the poles are given by

$$s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{\xi^2 - 1}$$

(b) If real parts of poles are positive, the poles lie on the right half plane (RHP) of S-plane.

(c) This is an unstable condition.

(vi) Case 6: ξ = -1

(a) The poles are given by s_1, $s_2 = \xi w_n, \xi w_n$

(b) The poles are located in the RHP of S-plane.

(c) The system is unstable.

(vii)Case 7: ξ < - 1

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

(a) The poles are given by

(b) The poles are located in the RHP of S-plane.

(c) The system is unstable

Figure 5.15 shows the location of poles for a second-order system for above cases

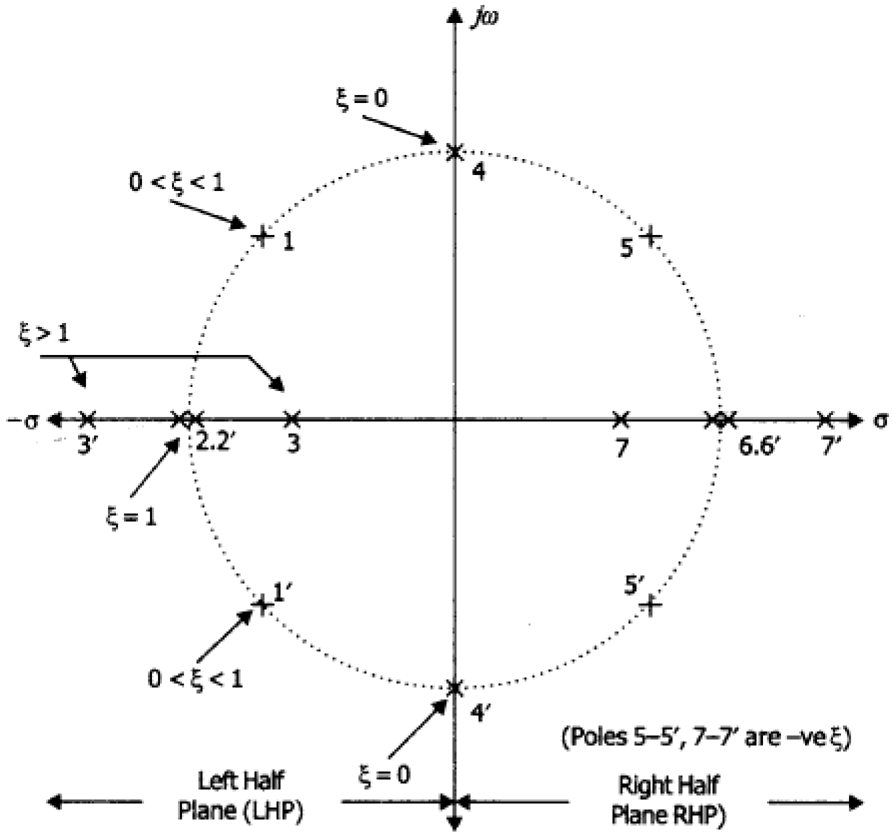

Fig. 5.15 Effect of ξ on pole location

Figure 5.16 shows ξ and nature of response.

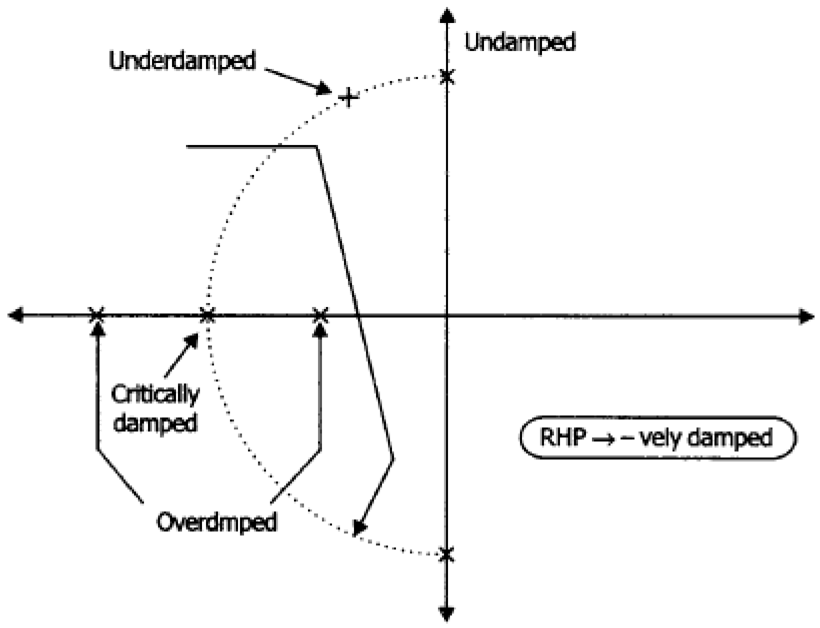

Fig. 5.16 ξ and nature of response

5.7.2 Effect of ξ on Nature of Response

Let us obtain the solution for response c (t) to a unit step input [r(t) = u(t)].

The Laplace transform of r(t) is given by

$$R(s) = \frac{1}{s} \tag{5.27}$$

Using Eq. (5.25), we get, from Eq (5.27),

$$C(s) = \left(\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\right) R(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} \tag{5.28}$$

Case 1: Underdamped Case ($0 < \xi < 1$)

From Eq (5.28) it can be written as

$$C(s) = \frac{1}{s} - \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= 1 - e^{-\xi\omega_n t}\left\{\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right\} = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \alpha) \tag{5.29}$$

Where $\omega_d^2 = \omega_n^2(1 - \xi^2)$. $\cos\alpha = \xi$ $\sin\alpha = \sqrt{1-\xi^2}$ $\alpha = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$ (5.30)

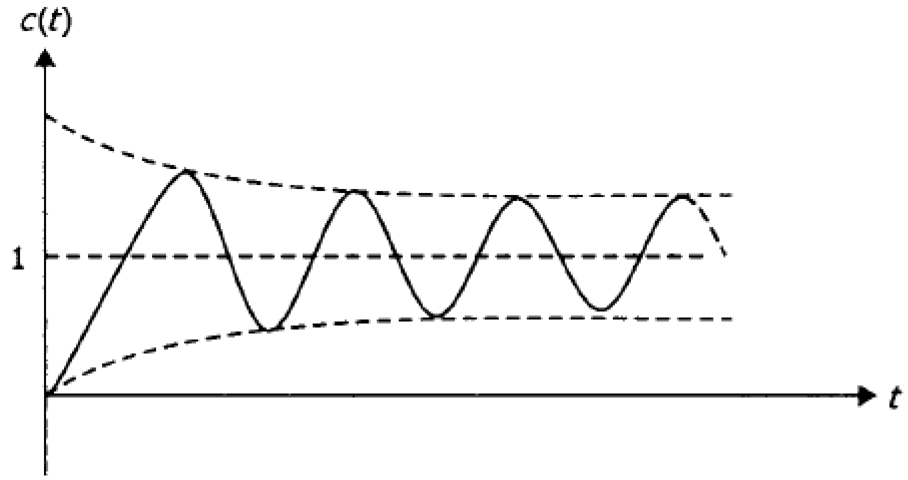

Fig. 5.17 Underdamped

Case 2: Critically Damped ($\xi = 1$)

For critically damping the roots are real and equal. Substituting $\xi = 1$ in Eq. (5.28), we get

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\omega_n s + \omega_n^2)} = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

$$c(t) = 1 - \omega_n e^{-\omega_n t} t - e^{-\omega_n t} = 1 - (1+\omega_n t) e^{-\omega_n t} \qquad (5.31)$$

From Fig. 8.21 depicts ***c(t)*** versus ***t*** where ***c(t)*** has been given in equation (5.31).

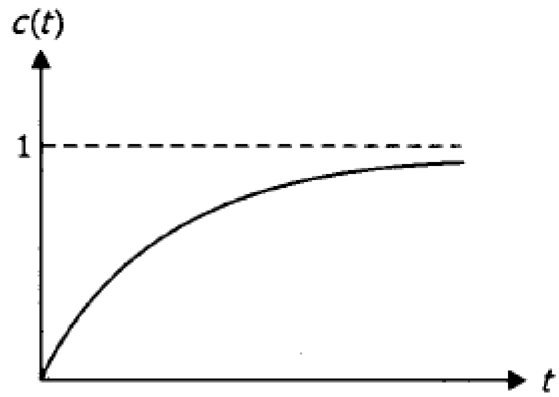

Fig. 5.18 Critically damped

Case 3: Overdamped Case (ξ >1)

From Eq. (5.28),

$$c(t) = 1 - \left[Ae^{-\left(\xi\omega_n + \omega_n\sqrt{\xi^2-1}\right)t} - e^{-\left(\xi\omega_n - \omega_n\sqrt{\xi^2-1}\right)t} \right]$$

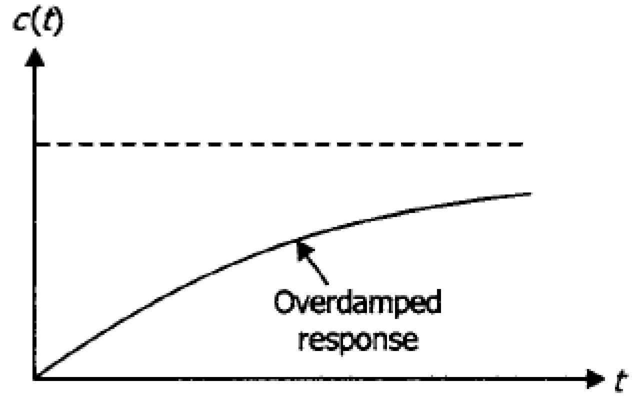

Fig. 5.19 Overdamped

(iv) Case 4: Undamped ($\xi = 0$)

Substituting $\xi = 0$ in Eq. (5.28), we get

$$C(s) = \frac{\omega_n^2}{s(s^2+\omega_n^2)} = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}$$

$$c(t) = 1 - \cos \omega_n t \qquad (5.32)$$

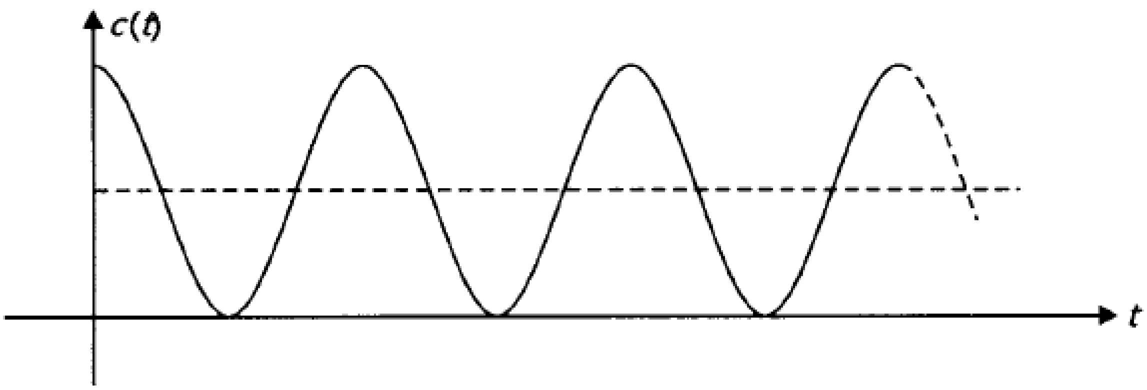

Fig. 5.20 Undamped

Case 5: Negative Damping ($0 > \xi > -1$)

From Eq. (5.28)

$$C(s) = \frac{\omega_n^2}{s(s^2+2\xi\,\omega_n s+\omega_n^2)} = \frac{\omega_n^2}{s\left[(s-a)^2+b^2\right]} = \frac{A}{s} + \frac{Bs+C}{(s-a)^2+b^2}$$

where the poles are 0 , $a \pm jb$ (a , $b > 0$).

$$c(t) = A + Be^{at}\cos bt + Ce^{at}\sin bt. \qquad (5.33)$$

Figure 5.21 shows the plot of c(t) versus t

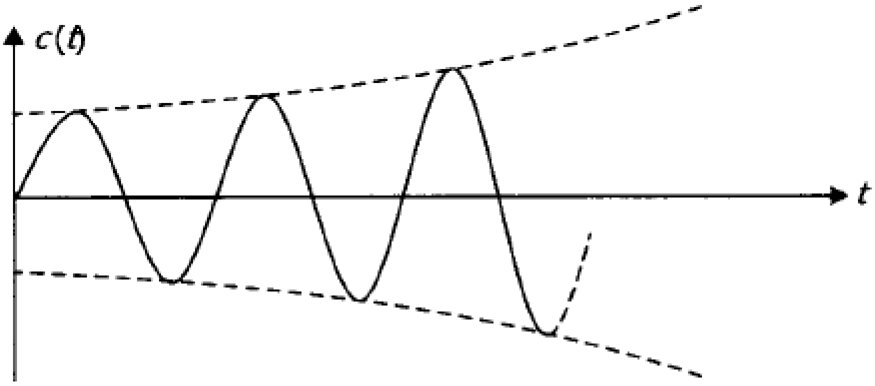

Fig. 5.21 Negative damping (0 > ξ > - 1)

Case 6: Negative Damping (ξ = -1)

From Eq. (5.28), it can be written as

$$C(s) = \frac{\omega_n^2}{s(s^2+2\omega_n s+\omega_n^2)} = \frac{\omega_n^2}{s(s-\omega_n)^2} = \frac{1}{s} + \frac{\omega_n}{s(s-\omega_n)^2} - \frac{1}{s-\omega_n}$$

$$c(t) = 1+\omega_n t\, e^{\omega_n t} - e^{\omega_n t} = 1+(\omega_n t - 1)\, e^{\omega_n t} \qquad (5.34)$$

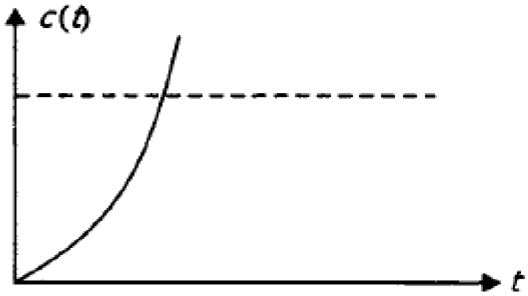

Fig. 5.22 Negative Damping (ξ = -1)

Case 7: Negative Damping (ξ < -1)

From Eq. (5.28),

$$C(s) = \frac{\omega_n^2}{s(s^2+2\xi\,\omega_n s+\omega_n^2)} = \frac{\omega_n^2}{s(s-a)(s-b)} = \frac{A}{s} + \frac{B}{s-a} + \frac{C}{s-b}$$

$$c(t) = A + B e^{at} + C e^{bt}$$

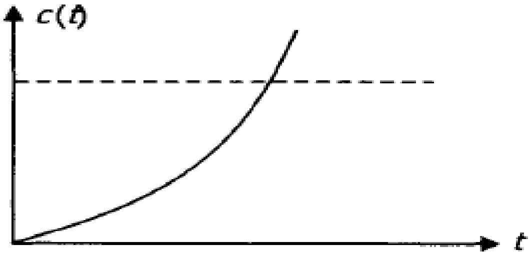

Fig. 5.23 Negative damping (ξ < - 1)

5.8 Derivation of Time Response Specifications

Figure 5.2 shows the plot of c(t) versus t for unit step input and different transient response specifications have also been pointed in Figure 5.2.

5.8.1 Delay Time (T_d)

The time required to reach 50% of output is known as delay time.

$$c(t) = \frac{1}{2} \text{ at } t = T_d$$

$$T_d = \frac{1+0.7\xi}{\omega_n} \qquad (5.35)$$

5.8.2 Rise Time (T_r)

Since the system is underdamped, therefore,

c(t) = 1 at t = T_r

From Eq (5.29),

$$T_r = \frac{\pi - \alpha}{\omega_d} \qquad (5.36)$$

5.8.3 Peak Time (T_p)

$$T_p = \frac{n\pi}{\omega_d} \qquad (5.37)$$

The first over short is obtained for n = 1 and the second overshoot is obtained for n = 2.

5.8.4 Peak Overshoot (M_p)

$$M_p = \frac{e^{-\xi\omega_n T_P}}{\sqrt{1-\xi^2}} \sin\alpha \qquad (5.38)$$

$$\% M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} \times 100 \qquad (5.39)$$

5.8.5 Settling Time (T_s)

For a 2% criterion,

$$T_s = 4T$$

$$T_s = \frac{4}{\xi\omega_n} \qquad (5.40)$$

$$\frac{1}{\xi\omega_n} = T = \text{time constant}$$

Sl. No.	Time Specifications	
	Type	**Formula**
1	Delay time	$t_d = \frac{1+0.7\zeta}{w_n}$
2	Rise time	$t_r = \frac{\pi - \varphi}{w_d}$
3	Peak time	$t_p = \frac{\pi}{w_d}$
4	Maximum overshoot	$M_p(\%) = 100 \times e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$
5	Settling time	$t_s = \frac{4}{\zeta w_n}$

Example 5.2:

A system has $G(s) = 20/(s^2 + 5s + 5)$ and unity feedback. Find (i) ω_n, (ii) ξ, (iii) ω_d, (iv) T_d, (v) T_r, (vi) T_p, (vii) M_p, and (viii) T_s.

Solution

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{20}{s^2+5s+5}}{1+\frac{20}{s^2+5s+5}} = \frac{20}{s^2+5s+25}$$

(i) Since the numerator may not be ω_n^2 due to the existence of a constant term in denominator of $G(s)$, ω_n^2 must be compared with the denominator of $C(s)/R(s)$ only.

$\therefore$ $\omega_n^2 = 25$ or $\omega_n = 5$ rad/sec

(ii) $2\xi\omega_n = 5$ or $\xi = \frac{5}{2\omega_n} = \frac{5}{2\times 5} = 0.5$

(iii) $\omega_d = \omega_n\sqrt{1-\xi^2} = 5\times\sqrt{1-0.5^2} = 4.33$ rad/sec

(iv) $T_d = \frac{1+0.7\xi}{\omega_n} = \frac{1+0.7\times 0.5}{5} = 0.27$ sec

(v) $T_r = \frac{\pi-\alpha}{\omega_d}$

$$\alpha = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \tan^{-1}\frac{\sqrt{1-0.5^2}}{0.5} = 1.0467 \text{ rad}$$

$\therefore$ $T_r = \frac{\pi-1.0467}{4.33} = 0.4834$ sec

(vi) $T_p = \frac{\pi}{\omega_d} = \frac{3.14}{4.33} = 0.725$ sec

(vii) $M_p = 100\times e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} = 100\times e^{-\frac{0.5\times 3.14}{\sqrt{1-0.5^2}}} = 16.32\%$

(viii) $T_s = \frac{4}{\xi\omega_n} = \frac{4}{0.5\times 5} = 1.6$ sec