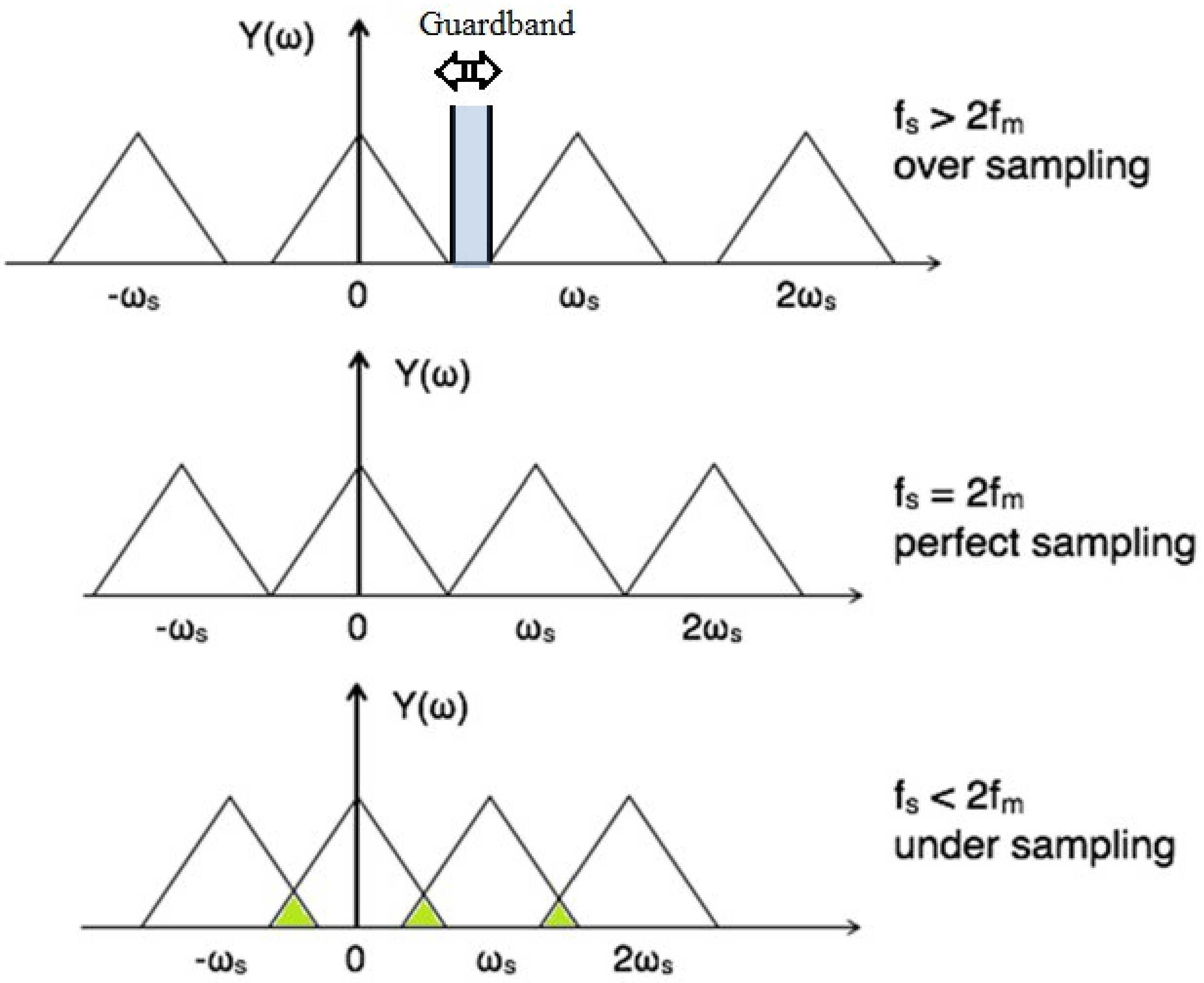
Sampling Theorem

The sampling rate higher than highest frequency of modulating signal creates a **guard band** between sampled spectrum . Therefore, a more practical LPF can be used to restore the modulating signal.

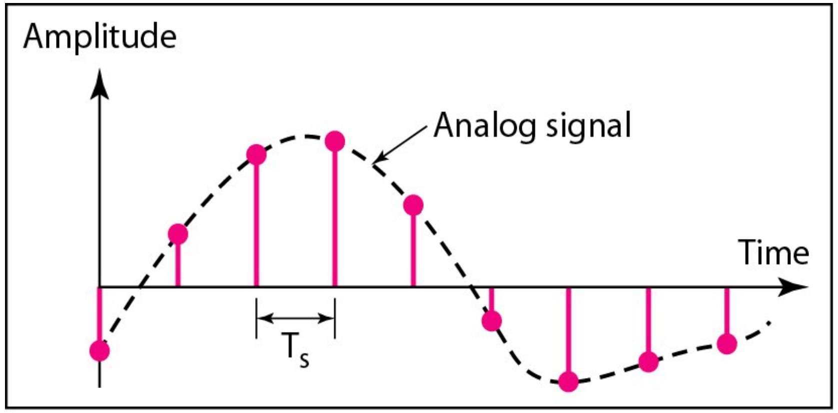


The overlapped region in case of under sampling represents aliasing effect, which can be removed by considering $f_s > 2f_m$

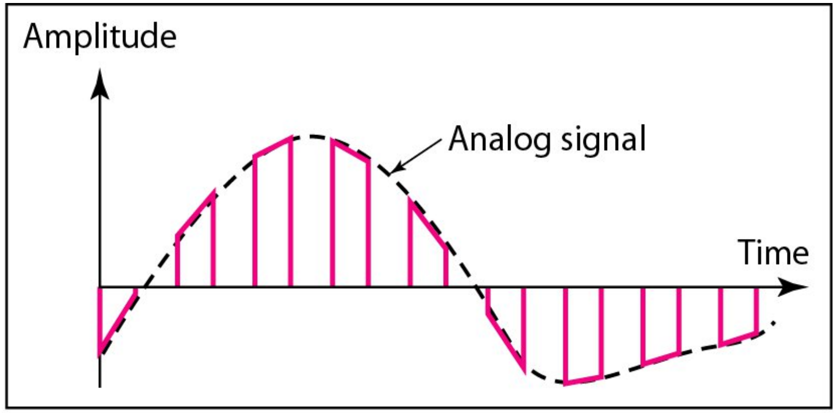
Sampling Types

There are three sampling methods:

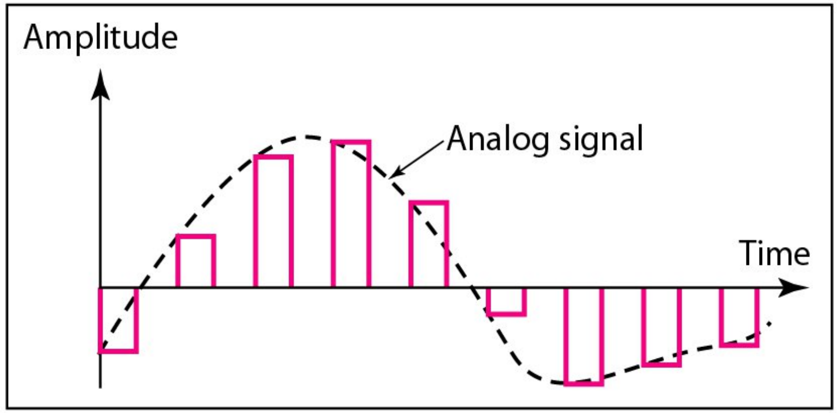
- **Ideal** - an impulse at each sampling instant
- **Natural** - a pulse of short width with varying amplitude
- **Flat top** - sample and hold, like natural but with single amplitude value



a. Ideal sampling



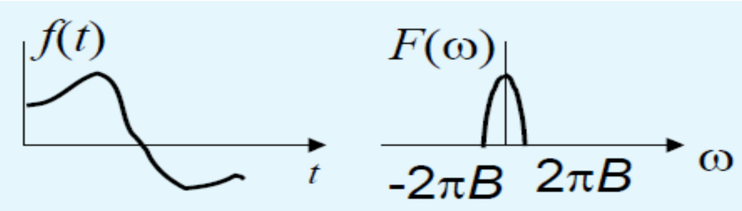
b. Natural sampling



c. Flat-top sampling

Proof

Consider a band-limited signal $f(t)$ having no spectral components above $f_m = B$ Hz.

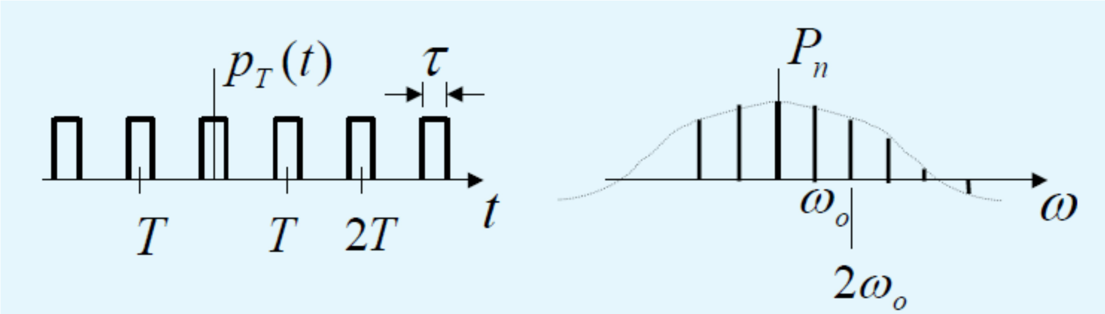


The signal is sampled using the periodic gate function $p_T(t)$. As $p_T(t)$ is a periodic signal, it can be represented by a Fourier series.

$$p_T(t) = \sum_{n=-\infty}^{\infty} P_n e^{jn\omega_o t} \qquad \omega_o = 2\pi / T$$

$$P_n = \frac{\tau}{T} Sa(n\pi\tau / T)$$

$$\boxed{Sa(x) = \frac{\sin x}{x}}$$



The sampled signal $f_s(t)$ is

$$f_s(t) = f(t)\, p_T(t)$$

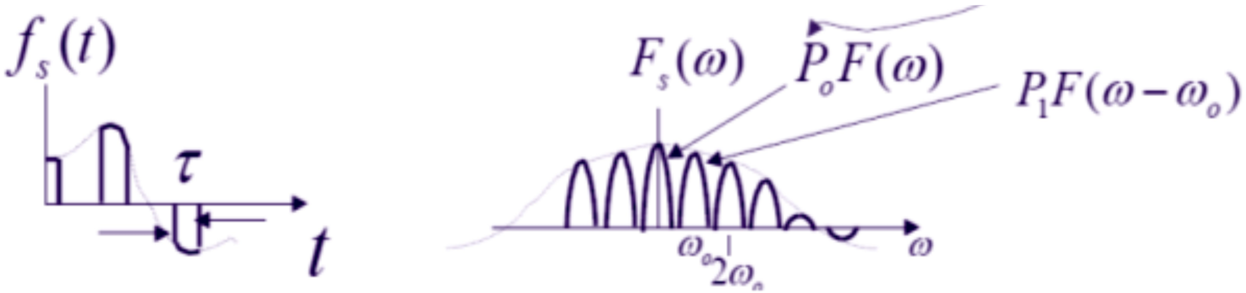
$$= f(t) \sum_{n=-\infty}^{\infty} P_n e^{jn\omega_o t}$$

Taking the Fourier transform, we have

$$F_s(\omega) = F\left\{ f(t) \sum_{n=-\infty}^{\infty} P_n e^{jn\omega_o t} \right\}$$

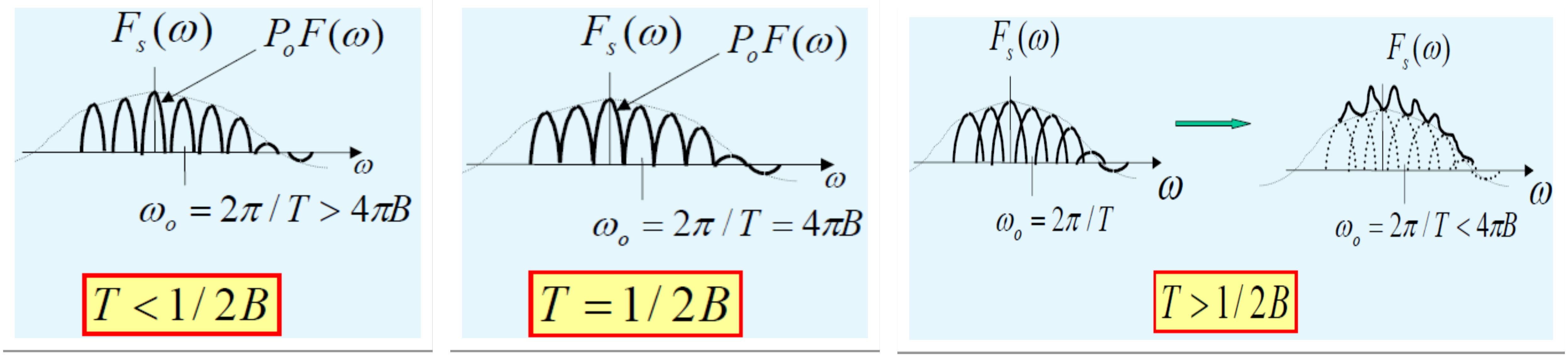
$$= \sum_{n=-\infty}^{\infty} P_n F\left\{ f(t) e^{jn\omega_o t} \right\} \qquad \text{(Linearity}$$

$$= \sum_{n=-\infty}^{\infty} P_n F(\omega - n\omega_o) \qquad \text{(frequency}$$



Therefore, the spectral density of the sampled signal is, within a constant factor, exactly the same as that of . In addition, it repeats itself periodically. The spectral density of the original signal can be retrieved by using a LPF on $F_s(\omega$.

However, if the sampling period $T > 1/2B$ ($f_s < \mathbf{2}\ f_m$) the replicas of will overlap and we cannot retrieve $F(\omega)$ from $F_s(\omega)$



$f_s > \mathbf{2}\ f_m$)

$f_s = \mathbf{2}\ f_m$)

$f_s < \mathbf{2}\ f_m$)

Example What is the minimum sampling frequency required for sampling a 10kHz sinusoid signal?

$$f_s \text{ min} \geq 2\ f_m = 20 \text{ kHz}$$

Example What is the minimum sampling frequency required for sampling a voice signal (0-3.3) kHz sinusoid signal?

$$f_s \text{ min} \geq 2\ f_m = 6.6 \text{ kHz}$$

In practice sampling frequency is **8 kHz**

Example What is the minimum sampling frequency required for sampling audio signal with (0-20) kHz sinusoid signal?

$$f_s \text{ min} \geq 2\ f_m = 40 \text{ kHz}$$

In practice sampling frequency for encoding music into CD is **44.1 kHz**

Example Determine the minimum sampling rate(Nyquist rate) for CT signal

$$s(t) = 5\cos(50\pi t) + 20\sin(300\pi t) - 10\cos(100\pi t)$$

Solution $f_1 = w_1/2\pi = 50\pi/2\pi = 25$ Hz, $f_2 = w_2/2\pi = 300\pi/2\pi = 150$Hz

$f_3 = w_3/2\pi = 100\pi/2\pi = 50$Hz;

Here, highest frequency component=150Hz

Hence **Nyquist rate**=2*150Hz=**300Hz**

Note: If signal has more than one frequency, the highest frequency is taken when considering sampling

Example : find the Nyquist rate and Nyquist interval of the continuous-time signal given below

$$\boldsymbol{s(t) = \frac{1}{2\pi} \cos(4000\pi t) \cos(1000\pi t)}$$

Solution: Signal can be written as

$$s(t) = \frac{1}{2\pi}\frac{1}{2}[\cos(4000\pi t + 1000\pi t) + \cos(4000\pi t - 1000\pi t)]$$

$$s(t) = \frac{1}{4\pi}[\cos(5000\pi t) + \cos(3000\pi t) = A1\ Cost(\omega m1) + A2\ Cos(\omega m2)$$

$$f1 = \frac{\omega m1}{2\pi} = \frac{5000\pi}{2\pi} = 2500\ Hz \qquad f2 = \frac{\omega m2}{2\pi} = \frac{3000\pi}{2\pi} = 1500\ Hz$$

The heights frequency of given Signal can be written as

$$fmax = Max[f1, f2] = 2500\ Hz$$

Then Nyquist rate = $2 \times fmax$= $2 \times 2500 = 5000\ Hz$

Nyquist interval= $\frac{1}{\text{Nyquist rate}} = \frac{1}{5000} = 2 \times 10{-}4 = 200\ miliseconds$

Question: Calculate the Nyquist rate in rad/sec and in Hz.

$$m(t) = 2\underbrace{\sin 4\pi t}_{A}.\underbrace{\cos 2\pi t}_{B}$$

$\omega_s = 2\omega_m \qquad f_s = 2f_m = \omega_s/2\pi$

$m(t) = x_1(t) + x_2(t)$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$m(t) = \sin(4\pi t + 2\pi t) + \sin(4\pi t - 2\pi t)$

$m(t) = \sin 6\pi t + \sin 2\pi t$

$\omega_1 = 6\pi \qquad \omega_2 = 2\pi \Rightarrow \omega_1 > \omega_2 \Rightarrow \omega_m = \omega_1 = 6\pi$

$\omega_s = 2\omega_m = 12\pi$ rad/sec

$$f_s = \omega_s/2\pi = \frac{12\pi \text{ rad/sec}}{2\pi \text{ rad}} = 6\text{ Hz}$$

$\Rightarrow m(t) = m_1(t) \cdot m_2(t)$

$\omega_s = \omega_{s_1} + \omega_{s_2}$

$\sin 4\pi t \Rightarrow \omega_{m_1} = 4\pi$

$\cos 2\pi t \Rightarrow \omega_{m_2} = 2\pi$

$\omega_{s_1} = 8\pi$

$\omega_{s_2} = 4\pi$

$\omega_s = 8\pi + 4\pi$

$\omega_s = 12\pi \text{ rad/sec}$

Homework: $m(t) = \cos 200\pi t.\cos 100\pi t$

Find the Nyquist rate in Hz.

Question: Let $x(t)$ be a signal with Nyquist rate ω_s. Determine the Nyquist rate for each of the following signals:

1. $x(t) + x(t-1)$
2. $dx(t)/dt$
3. $x^2(t)$
4. $x(t)\cos(\omega_s t)$

Solution: $x(t) \longrightarrow NR = \omega_s$

$$\underset{\omega_s}{x(t)} \xrightarrow{T.S.} \underset{\omega_s}{x(t-1)}$$

1. $m(t) = x(t) + x(t-1) \longrightarrow NR = ?$

$$\underset{NR = \omega_s}{m(t)} \qquad x(t) \to \omega_s \qquad x(t-1) \to \omega_s$$

2. $m(t) = \dfrac{d\,x(t)}{dt}$ $\quad x(t) \to \omega_s$ $\longrightarrow \omega_s$

3. $m(t) = x^2(t) = [x(t)]^{2}$

$x(t) \longrightarrow \omega_s$

$[x(t)]^n \longrightarrow n \times \omega_s$

$NR = 2 \times \omega_s$

$NR = 2\omega_s$ Ans

4. $m(t) = \underbrace{x(t)}_{m_1(t)} \cdot \underbrace{\cos(\omega_s t)}_{m_2(t)}$

$$NR = \cancel{\omega_{s1}} + \cancel{\omega_{s2}} = 2\cancel{\omega_m}\,\omega_s = 3\omega_s \text{ Ans}$$

$$2\omega_s + 2\omega_s$$

Pulse Analog Modulation : Pulse Amplitude Modulation (PAM)

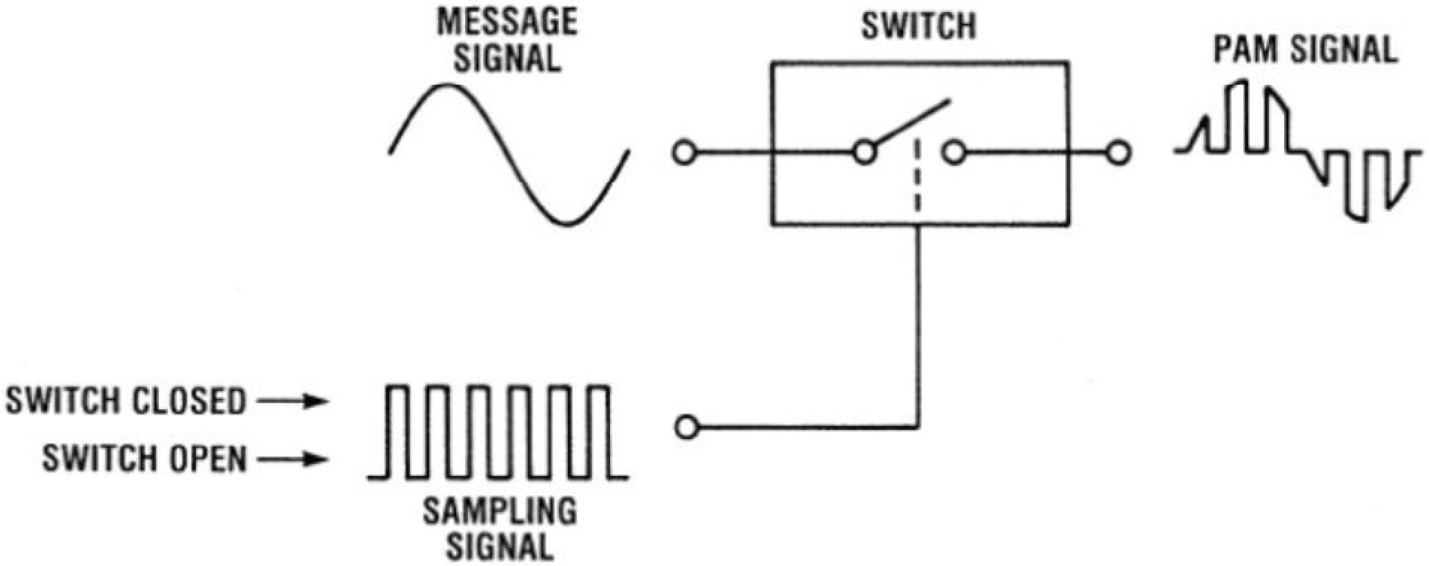
Pulse-amplitude modulation (PAM), is the simplest form of analog pulse modulation. **PAM** is a form of signal modulation where the message information is encoded in the amplitude of a series of signal pulse. It is an analog pulse modulation scheme in which the amplitudes of a train of carrier pulses are varied according to the sample value of the message signal. **Demodulation** is performed by detecting the amplitude level of the carrier at every single period by using **low pass filter.**

- 1. **Instantaneous sampling** of the message signal ***m(t)*** every T_s seconds, where the sampling rate $f_s = 1/T_s$ is chosen in accordance with the sampling theorem.
- 2. **Lengthening or Holding** the duration of each sample so obtained to some constant value ***T***. These two operations are jointly called **"sample and hold."**
- The PAM signal is then sent by either wire or cable or it is used to modulate a carrier.

There are two classes of PAM signals: PAM that uses **natural sampling** (gating) and PAM that uses **instantaneous sampling** to produce a flat-top pulse. The flat-top type is more useful for conversion to PCM; however, the naturally sampled type is easier to generate and is used in other applications.

Generation of PAM

A PAM signal is generated by using a pulse train, called the sampling signal (or clock signal) to operate an electronic switch or "chopper". This produces samples of the analog message signal



Natural sampling – generation of PAM signals.

The switch is closed for the duration of each pulse allowing the message signal at that sampling time to become part of the output. The switch is open for the remainder of each sampling period making the output zero. This type of sampling used here is the **natural sampling.**