The **frequency-shifting / modulation** property is one of the most important Fourier transform properties as ***modulation*** is the basic operation that underlies all communication systems such as *Amplitude Modulation* (**AM**) and *Frequency Modulation* (**FM**).

Let $x(t)$ be a waveform with the Fourier transform $X(f)$. Then

$$\boxed{x(t)\,e^{j2\pi f_0 t} \iff X(f-f_0)}$$

Using the frequency-shifting property together with the linearity of the Fourier transform we can express the effect of multiplying a signal with a sinusoid:

$$\begin{aligned}
\mathcal{F}\left[\,x(t)\cos 2\pi f_0 t\,\right] &= \mathcal{F}\left[\frac{1}{2}x(t)e^{j2\pi f_0 t} + \frac{1}{2}x(t)e^{-j2\pi f_0 t}\right] \\
&= \mathcal{F}\left[\frac{1}{2}x(t)e^{j2\pi f_0 t}\right] + \mathcal{F}\left[\frac{1}{2}x(t)e^{-j2\pi f_0 t}\right] \\
&= \frac{1}{2}\left[X(f-f_0) + X(f+f_0)\right]
\end{aligned}$$

And similarly

$$\mathcal{F}\left[\,x(t)\sin 2\pi f_0 t\,\right] = \mathcal{F}\left[\frac{1}{2j}x(t)e^{j2\pi f_0 t} - \frac{1}{2j}x(t)e^{-j2\pi f_0 t}\right]$$

$$= \mathcal{F}\left[\frac{1}{2j}x(t)e^{j2\pi f_0 t}\right] - \mathcal{F}\left[\frac{1}{2j}x(t)e^{-j2\pi f_0 t}\right]$$

$$= \frac{1}{2j}\left[\,X(f-f_0) - X(f+f_0)\,\right]$$

$$= \frac{1}{2}\left[\,X(f-f_0)e^{-j\pi/2} + X(f+f_0)e^{j\pi/2}\,\right]$$

Since:

- $e^{j\frac{\pi}{2}} = j$
- $e^{-j\frac{\pi}{2}} = -j$

Hence

$$x(t)\,\cos 2\pi f_0 t \quad \Longleftrightarrow \quad \frac{1}{2}\left[\,X(f-f_0) + X(f+f_0)\,\right]$$

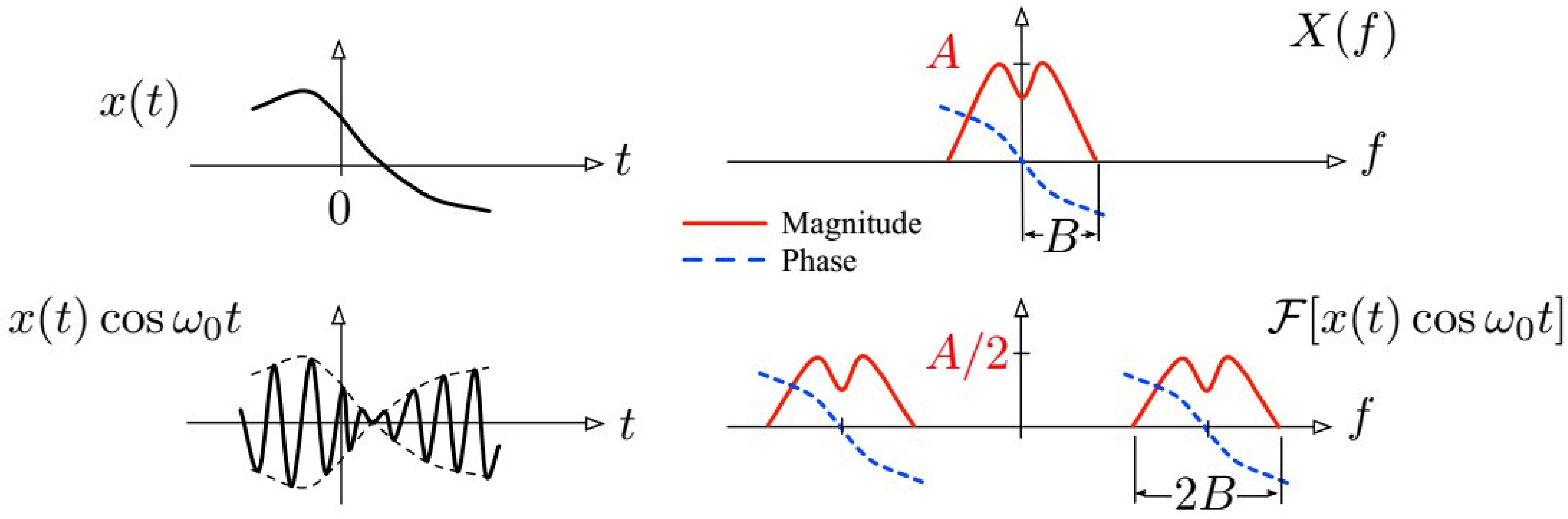
$$x(t)\,\sin 2\pi f_0 t \quad \Longleftrightarrow \quad \frac{1}{2}\left[\,X(f-f_0)e^{-j\pi/2} + X(f+f_0)e^{j\pi/2}\,\right]$$

Frequency-Shifting Property

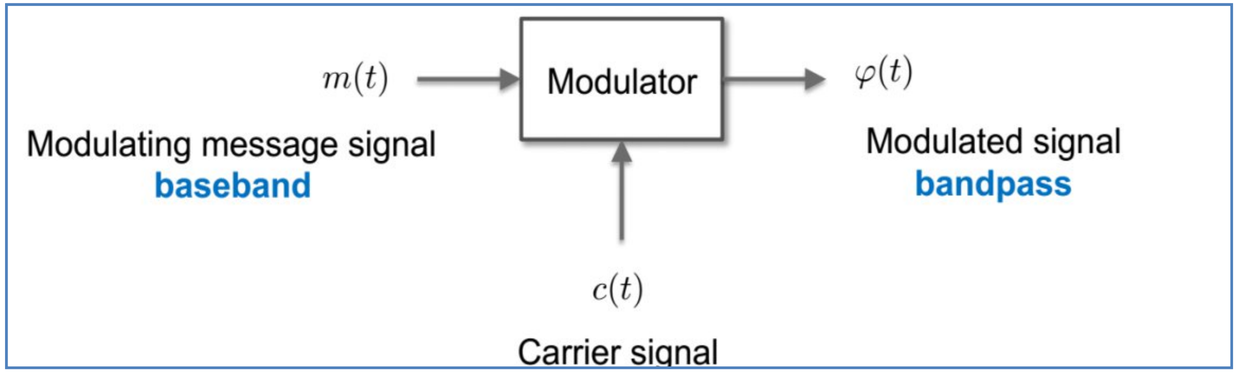
Multiplying a signal by a sinusoid moves the signal spectrum up and down the frequency band. This operation is known as **modulation**.

Example

$$x(t)\,\cos 2\pi f_0 t \quad \Longleftrightarrow \quad \frac{1}{2}\left[X(f-f_0)+X(f+f_0)\right]$$



Basic Principles of Amplitude Modulation



$$\boxed{\varphi(t) = a(t) \cos \theta(t)}$$

(possibly) time-varying amplitude

(possibly) time-varying phase

$\varphi(t)$ represents a **rotating phasor** of:

- time-varying amplitude $a(t)$
- generalized phase $\theta(t)$
- instantaneous frequency $f_i(t)$:

$$\boxed{f_i(t) = \frac{1}{2\pi} \frac{d}{dt} \theta(t)}$$

Equivalently, **phase** is the time integral of **frequency**.

Basic Principles of Amplitude Modulation

Amplitude Modulation (AM) is the process of changing the **amplitude** of a relatively high frequency **carrier signal** in proportion with the instantaneous value of the **modulating signal**. High frequency carrier signal is also termed the radio-frequency (RF) signal because it is at a high-enough frequency to be transmitted through free space as a radio wave. Low frequency information signal is also term such as modulating signal, intelligence or baseband signal.

$m(t)$ = modulating message signal
$c(t)$ = carrier signal
$\varphi(t)$ = modulated signal

Types of Amplitude Modulation (AM)

AM is itself divided into different types:

1. **Double Sideband with carrier (we will call it AM or (DSB-FC)).**

This is the most widely used type of AM modulation. In fact, all radio channels in the AM band use this type of modulation.

2. **Double Sideband Suppressed Carrier (DSB-SC).**

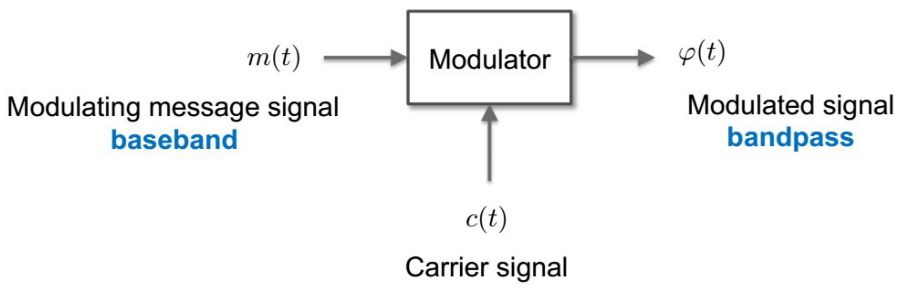
This is the same as the AM modulation above but without the carrier.

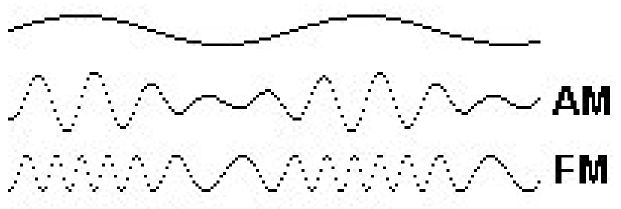
3. **Single Sideband (SSB).**

In this modulation, only half of the signal of the DSBSC is used.

4. **Vestigial Sideband (VSB).**

This is a modification of the SSB to ease the generation and reception of the signal.





AM is characterized by the fact that the amplitude **A** of the ***carrier*** $c(t) = A_c \cos(2\pi f_c t + \theta)$ is varied in proportion to the baseband (message) signal *m(t)*, the ***modulating signal.*** The frequency and the phase are constant. We can assume $\theta = 0$ without a loss of generality. Then the modulated signal is given by

$$\varphi_{\text{DSB-SC}}(t) = m(t)c(t) = A_c m(t) \cos 2\pi f_c t$$

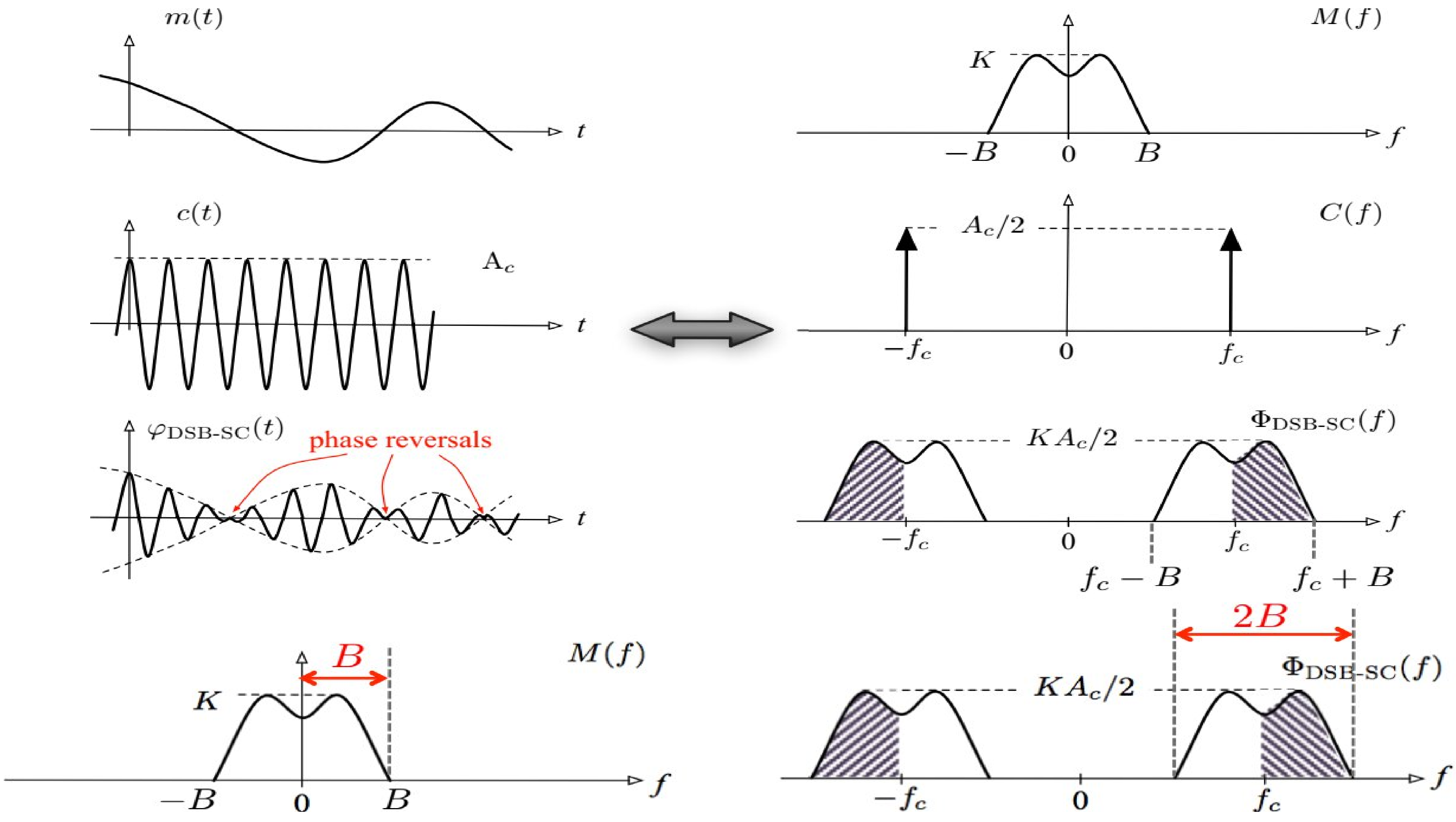
with spectrum:

$$\Phi_{\text{DSB-SC}}(f) = \frac{A_c}{2}\left[M(f - f_c) + M(f + f_c)\right]$$

This type of modulation simply shifts the spectrum of ***m(t)*** to the left and the right of the carrier frequency .

Note that the modulated signal spectrum centred at *fc* is composed of two parts: a portion that lies above *fc* known as the upper sideband (USB), and a portion that lies below *fc* , known as the lower sideband (LSB). Similarly, the spectrum centred at *-fc* has upper and lower sidebands. Hence, this is a modulation scheme with **double sidebands**.

It can be seen that the modulated signal in this scheme does not contain a discrete component of the carrier frequency . For this reason it is called **double sideband suppressed carrier** (DSB-SC) **modulation.**



Transmission Bandwidth

If the baseband message signal $m(t)$ has the bandwidth ***B***-Hz, then the DSB-SC signal $\varphi_{\text{DSB-SC}}(t)$ requires a transmission bandwidth of ***2B***-Hz

Synchronous or Coherent detection

The original signal X can be recovered from the amplitude modulated signal Y by multiplying by the carrier and then low-pass filtering.

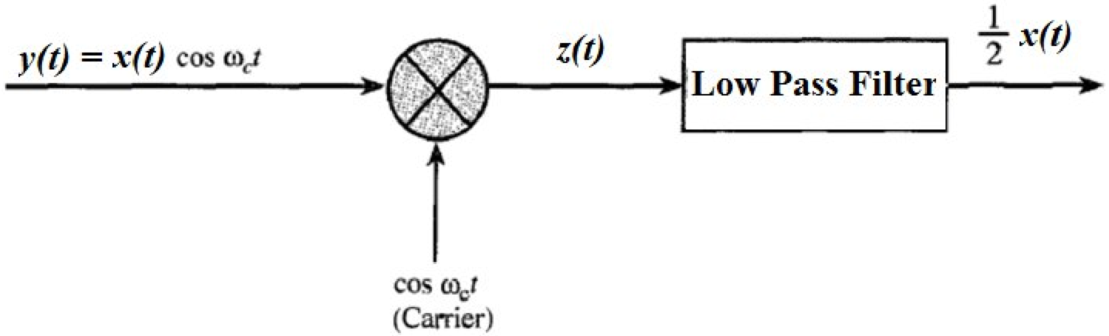
This process is called **synchronous demodulation**.

$$y(t) = x(t)\cos\omega_c t$$

$$z(t) = y(t)\cos\omega_c t$$

$$= x(t) \times \cos\omega_c t \times \cos\omega_c t$$

$$= x(t)\left(\frac{1}{2} + \frac{1}{2}\cos(2\omega_c t)\right)$$



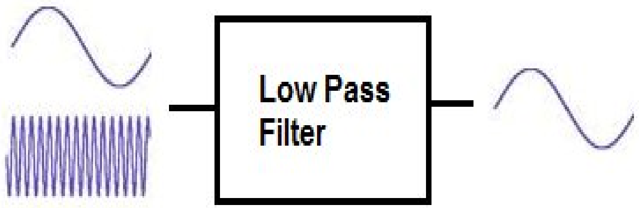
- Finding the Fourier transform of the signal z(t)

$$Z(f) = \frac{1}{2}X(f) + \frac{1}{4}[X(f+2f_c) + X(f-2f_c)]$$

- Signal $z(t)$ consists of two components $(1/2)x(t)$ and $(1/2)\, x(t)\cos(2w_c t)$, with their nonoverlapping spectra.
- The spectrum of the second component, being a modulated signal with carrier frequency $2f_c$, is centred at $\pm 2f_c$ and this component is suppressed by low-pass filter.

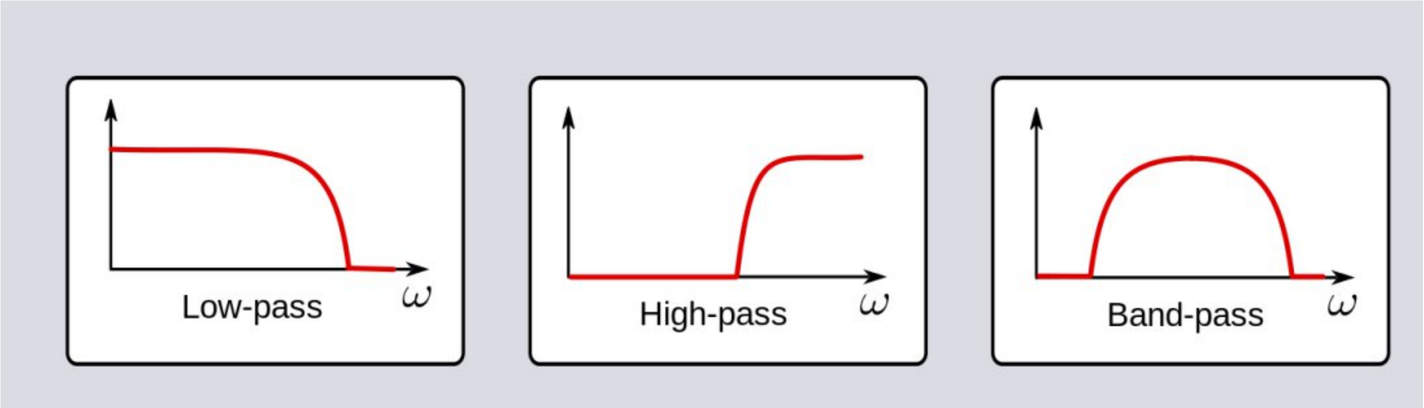
- On the other hand, the desired component $(1/2)X(f)$, being a low-pass spectrum (centred at $f = 0$) passes through the filter unharmed, resulting in (½)x(t).
- You can get rid of the inconvenient fraction ½ in the output by using a carrier $2\cos(w_c t)$ instead of $\cos(w_c t)$.

Low-pass filter – low frequencies are passed, high frequencies are attenuated.



High-pass filter – high frequencies are passed, low frequencies are attenuated.

Band-pass filter – only frequencies in a frequency band are passed.



The **DSB-SC** modulation translates the frequency spectrum to the left and the right by ***fc***(that is, at +***fc*** and -***fc***). To recover the original signal *m(t)* from the modulated signal, it is necessary to retranslate the spectrum to its original position. The demodulation consists of multiplication of the incoming modulated signal by a $Cos\ \omega_c t$ followed by a **low pass filter**.

If multiplying $m(t)$ with $c(t) = A_c \cos \omega_c t$, it shifts $M(f)$ to $\pm f_c$. What happens if we multiply $\varphi_{\text{DSB-SC}}(t)$ with $\cos \omega_c t$ again …

$$\begin{aligned}\varphi_{\text{DSB-SC}}(t)\cos\omega_c t &= A_c\, m(t)\cos^2\omega_c t,\\ &= \frac{A_c}{2}m(t) + \frac{A_c}{2}m(t)\cos 2\omega_c t.\end{aligned}$$

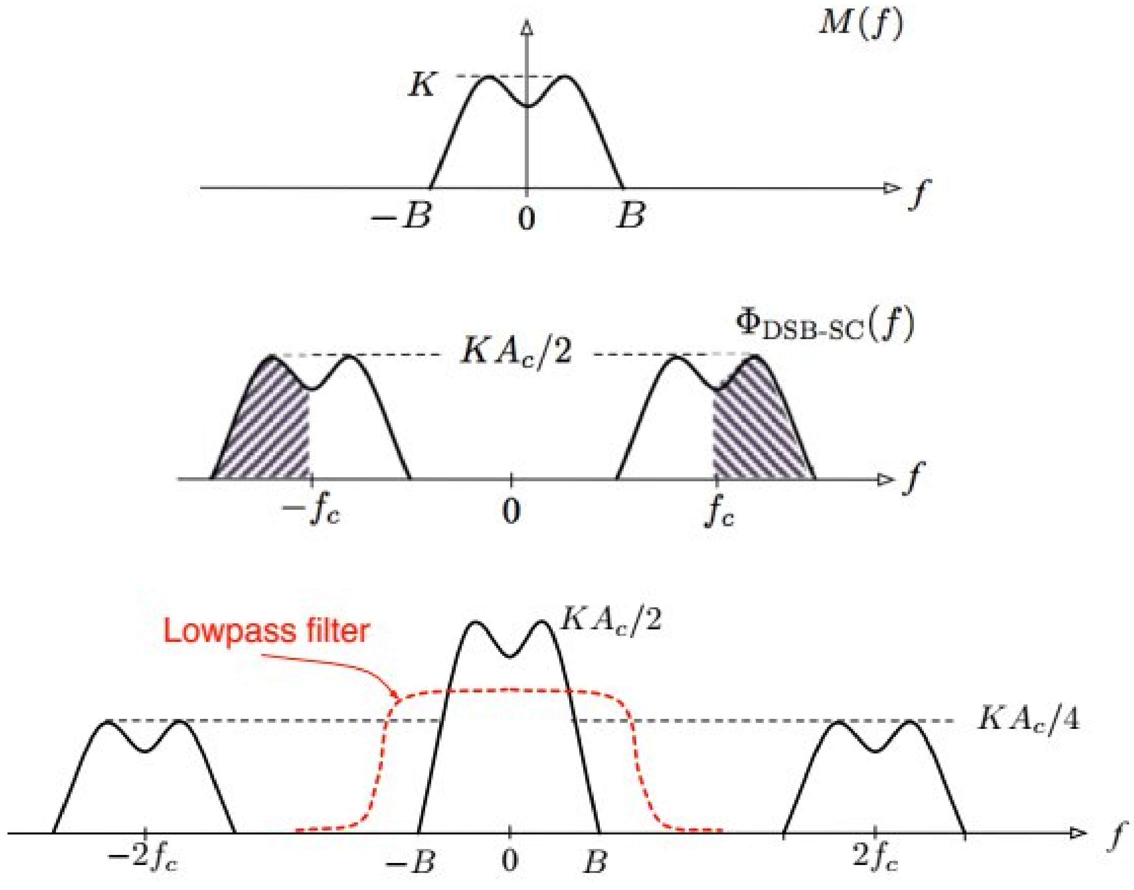
magnitude scaled **baseband** message signal

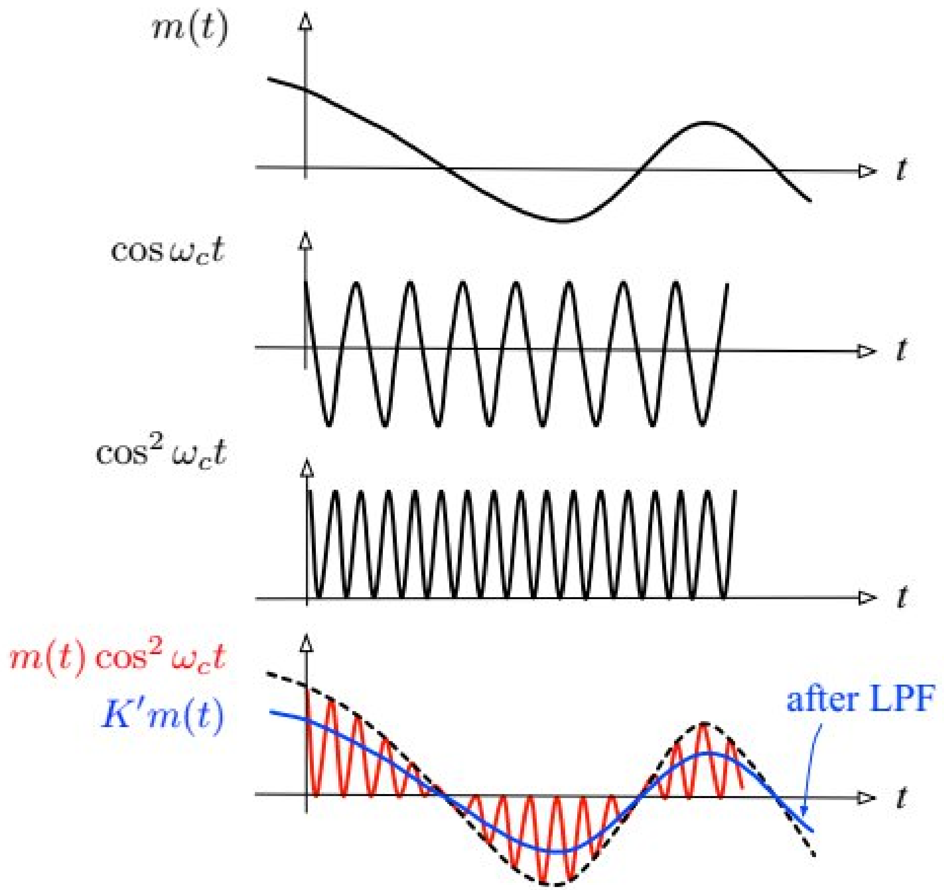
narrowband spectrum **centered at ± 2f_c**

Under the assumption that $f_c \gg B$ we can recover $m(t)$ from $\varphi_{\text{DSB-SC}}(t)\cos\omega_c t$ by **lowpass filtering**.

Note :$\cos^2(x) = 1/2 + 1/2\cos(2x)$

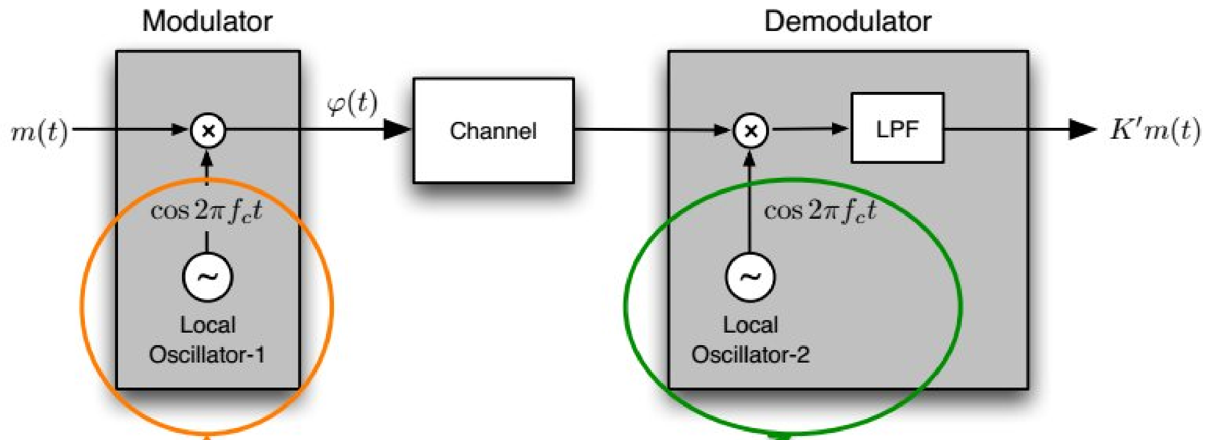
The demodulation process of a DSBSC signal involves obtaining the original information signal or scaled version of it from the modulated signal. This can be done by multiplying the modulated signal with another carrier signal that **has EXACTLY the same frequency and phase as the carrier signal in the modulator block**. The amplitude of the two carrier signals in the modulator and demodulator are not important since they just affect the magnitude of the different intermediate signals and final output signal of the demodulator.





DSB-SC: Modulation and demodulation

The complete **modulation** – **transmission** – **demodulation** chain for DSB-SC amplitude modulated signals:



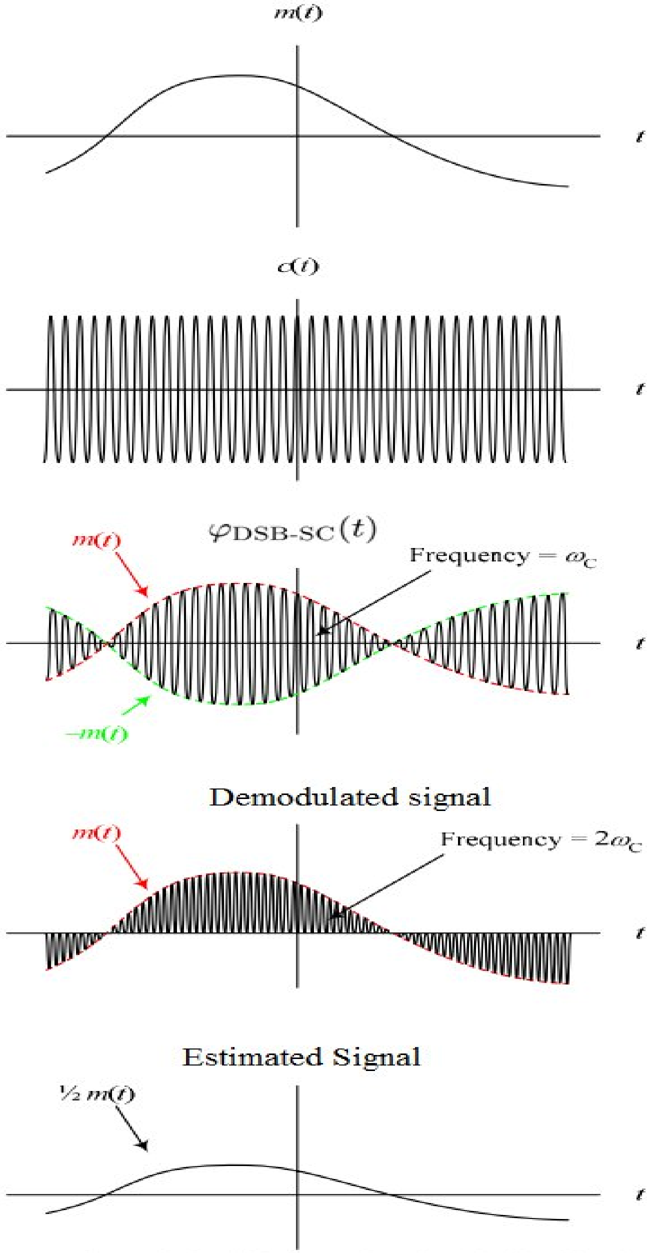
We assumed LO-1 in the TX and LO-2 in the RX are synchronized:

TX $\quad l_m(t) = \cos(2\pi f_1 t + \phi_1)$

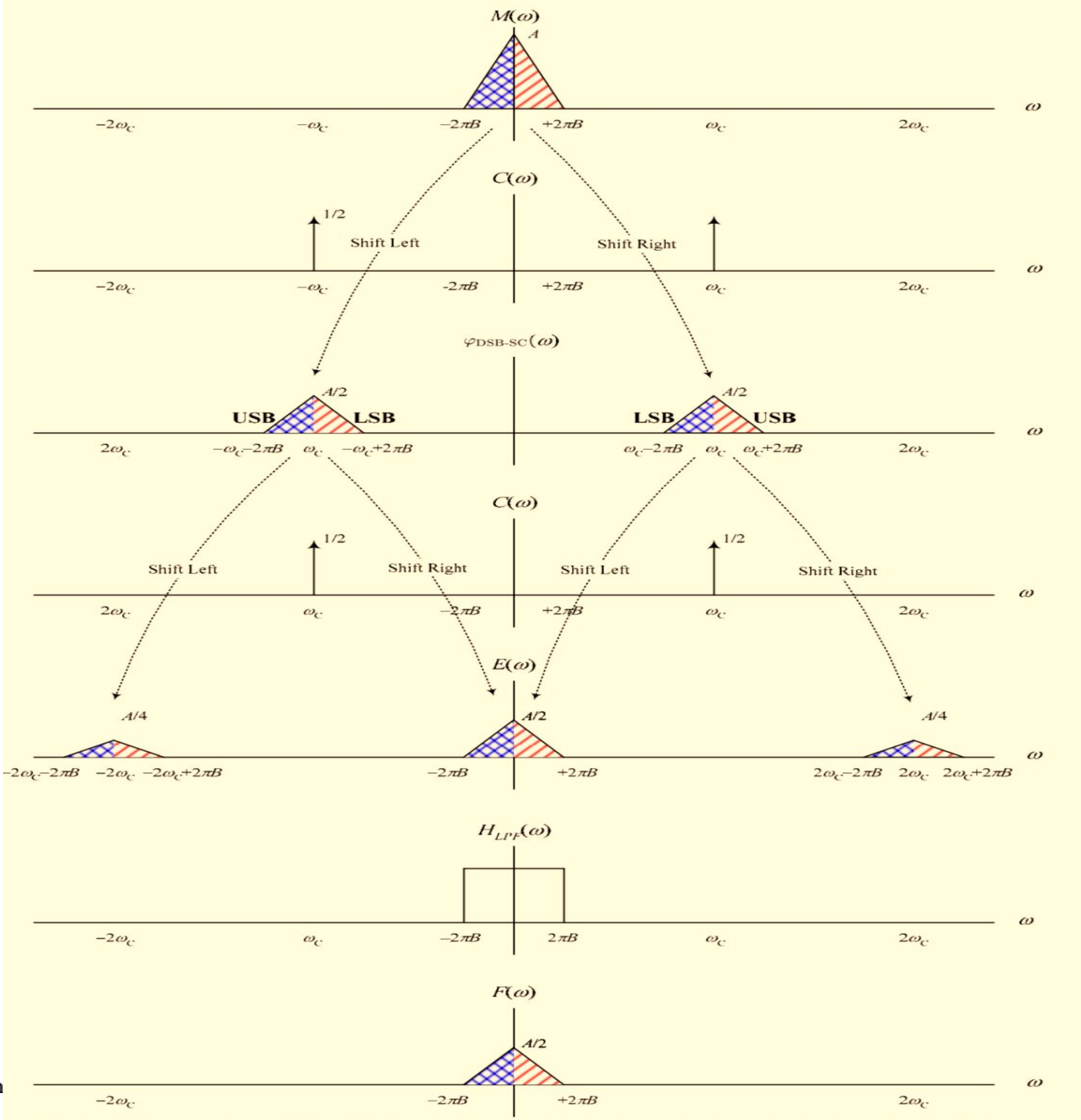
RX $\quad l_d(t) = \cos(2\pi f_2 t + \phi_2)$

with $\quad f_1 = f_2 = f_c \quad$ and $\quad \phi_1 = \phi_2$

AM DSB-SC Modulation and Demodulation:



Time–domain representation of the different signals obtained in the DSBSC modulation demodulation process.



Frequency–domain representation of the different signals obtained in the DSBSC modulation–demodulation process.

Example Given an audio signal $m(t) = 2\cos 1000\pi t + \cos 2000\pi t$ and a carrier of $c(t) = 10\cos 10^5 \pi t$.

Write down the expression for the upper sideband terms of the product signal for AM-DSB-SC.

Solution

$$\varphi_{DSB-SC}(t) = m(t)\ c(t)$$

$$\varphi_{DSB-SC}(t) = [2\cos(1000\pi t) + \cos(2000\pi t)]\ (10\cos\left(10^5\pi t\right))$$

$$\varphi_{DSB-SC}(t) = [20\cos\left(10^3\pi t\right)\cos\left(10^5\pi t\right) + 10\cos(2000\pi t)\cos\left(10^5\pi t\right)]$$

$$= 10(\cos 10.1\times 10^4\pi t + \cos 9.9\times 10^4\pi t) + 5(\cos 10.2\times 10^4\pi t + \cos 9.8\times 10^4\pi t)$$

The upper sideband term:

$$USB\ for\ AM-DSB-SC = 10(\cos 10.1\times 10^4\pi t) + 5(\cos 10.2\times 10^4\pi t)$$

Note : $\cos(x)\cos(y)=0.5\ [\cos(x-y)+\cos(x+y)]$

Note : When the modulating signal is a pure sinusoid or tone, *cos (2π* f_m *t)*, the modulation is called ***tone modulation***.

$$\varphi_{\text{DSB-SC}}(t) = m(t)c(t) = A_c m(t) \cos 2\pi f_c t$$

with spectrum:

$$\Phi_{\text{DSB-SC}}(f) = \frac{A_c}{2}\left[M(f - f_c) + M(f + f_c)\right]$$

Power of DSBSC is the sum of powers of the USB and the LSB components. $P_T = P_{\text{USB}} + P_{\text{LSB}}$

Power Ⓟ $\langle \Phi^2(t) \rangle = \frac{1}{2} A_c^2 \langle m^2(t) \rangle$

Power Efficiency (η) is Defined as the percentage of the total power of the modulated signal that conveys information to the total power = Signal Power/Total Power . The total power is the sum of the carrier power and the sideband power, i.e. The ratio of total sideband power to total input power gives the maximum transmission power efficiency.

Ⓟ $$\eta = \frac{\langle m^2(t) \rangle}{\langle m^2(t) \rangle} \times 100 = 100\%$$

Disadvantages of DSBSC:

- Less information about the carrier will be delivered to the receiver.
- Needs a coherent carrier detector at receiver