Multiplication $x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$

The multiplication property is the dual property of convolution and is referred to as the frequency convolution theorem. Thus, multiplication in time domain becomes convolution in the frequency domain.

Parseval's Relations for Fourier transform $\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$

Equation above is called Parseval's identity (or Parseval's theorem) for the Fourier transform. Note that the quantity on the left-hand side of this Eq. is the normalized energy content E of x(t). Parseval's identity says that this energy content ***E*** can be computed by integrating $\left|X(\omega)^2\right|$ over all frequencies ***ω.*** For this reason, $\left|X(\omega)^2\right|$ is often referred to as the ***energy-density spectrum*** of ***x(t)***, and Eq. above is also known as the energy theorem. Table below contains a summary of the properties of the Fourier transform presented in this section. Some common signals and their Fourier transforms are given in second Table.

Example : Obtain the F.T. of

$$x(t) = te^{-at}u(t)$$

from property of Fourier transform $\mathrm{FT}[tx(t)] = j\frac{d}{dw}X(w)$

$$\mathrm{FT}\left[e^{-at}\right] = \frac{1}{a+jw}$$

$$\mathrm{FT}(te^{-at}) = j\frac{d}{dw}\left(\frac{1}{a+jw}\right) = j\frac{(a+jw)\frac{d}{dw}(1) - 1\frac{d}{dw}(a+jw)}{(a+jw)^2} = \frac{1}{(a+jw)^2}$$

Common Fourier Transform Pair

$x(t)$	$X(\omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin\omega_0 t$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$u(-t)$	$\pi\delta(\omega)-\frac{1}{j\omega}$
$e^{-at}u(t), a>0$	$\frac{1}{j\omega+a}$
$te^{-at}u(t), a>0$	$\frac{1}{(j\omega+a)^2}$
$e^{-a\|t\|}, a>0$	$\frac{2a}{a^2+\omega^2}$
$\frac{1}{a^2+t^2}$	$e^{-a\|\omega\|}$
$e^{-at^2}, a>0$	$\sqrt{\frac{\pi}{a}}e^{-\omega^2/4a}$
$p_a(t)=\begin{cases}1 & \|t\|<a\\ 0 & \|t\|>a\end{cases}$	$2a\frac{\sin\omega a}{\omega a}$
$\frac{\sin at}{\pi t}$	$p_a(\omega)=\begin{cases}1 & \|\omega\|<a\\ 0 & \|\omega\|>a\end{cases}$
$\operatorname{sgn} t$	$\frac{2}{j\omega}$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0), \omega_0=\frac{2\pi}{T}$

Properties of Fourier Transform

Property	Signal	Fourier transform
	$x(t)$	$X(\omega)$
	$x_1(t)$	$X_1(\omega)$
	$x_2(t)$	$X_2(\omega)$
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{\omega}{a}\right)$
Time reversal	$x(-t)$	$X(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time differentiation	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Frequency differentiation	$(-jt)x(t)$	$\frac{dX(\omega)}{d\omega}$
Integration	$\int_{-\infty}^{t}x(\tau)\,d\tau$	$\pi X(0)\delta(\omega)+\frac{1}{j\omega}X(\omega)$
Convolution	$x_1(t)*x_2(t)$	$X_1(\omega)X_2(\omega)$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega)*X_2(\omega)$
Parseval's relations	$\int_{-\infty}^{\infty}\|x(t)\|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\|X(\omega)\|^2\,d\omega$	

We now consider the frequency domain representation of discrete-time signals that are not necessarily periodic. For continuous-time signals, we obtained such a representation by defining the Fourier transform of a signal ***x(t)***, as

$$X(\omega) = \mathscr{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

For discrete-time signals, we consider an analogous definition of the Fourier transform (**DTFT**)

$$X(\Omega) = \mathscr{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\ e^{-j\Omega\, n}$$

It is important to note here that ***X(Ω)*** is periodic with period 2π. According to above definition, it follows that

$$X(\Omega + 2\pi) = \sum_{-\infty}^{\infty} x[n]\ e^{-j(\Omega+2\pi)\, n} = \sum_{-\infty}^{\infty} x[n]\ e^{-j\Omega n} e^{-j2\pi n} = X(\Omega)$$

As a consequence, we have to consider values of **Ω** only over the range [0, 2π]. Furthermore, **Ω** is continuous, this fact makes the spectrum of X(**Ω**) continuous and periodic function of **Ω** with period 2π.

To find the inverse relation between ***X(Ω)*** and ***x[n]***, we replace the variable ***n*** by ***m*** to get

$$X(\Omega) = \sum_{m=-\infty}^{\infty} x[m]\; e^{-j\Omega\, m}$$

Now multiply both sides by $e^{j\Omega n}$ and integrate over the range [0, 2π] to get

$$\int_{2\pi} X(\Omega)\, e^{-j\Omega\, n} d\Omega = \int_{2\pi} \sum_{m=-\infty}^{\infty} x[m]\; e^{j\Omega\,(n-m)}\; d\Omega$$

Interchanging the orders of summation and integration then gives

$$\int_{2\pi} X(\Omega)\, e^{-j\Omega\, n} d\Omega = \sum_{m=-\infty}^{\infty} x[m] \int_{2\pi} e^{j\Omega\,(n-m)}\; d\Omega$$

It can be verified that $\int_{2\pi} e^{j\Omega(n-m)} d\Omega = \begin{cases} 2\pi, & n = m \\ 0, & n \neq m \end{cases}$

We can, therefore, write the inverse DTFT as

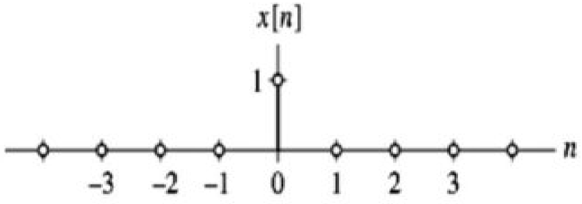
$$x[n] = \frac{1}{2\pi}\int_{0}^{2\pi} X(\Omega)\, e^{j\Omega n}\; d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n} d\Omega$$

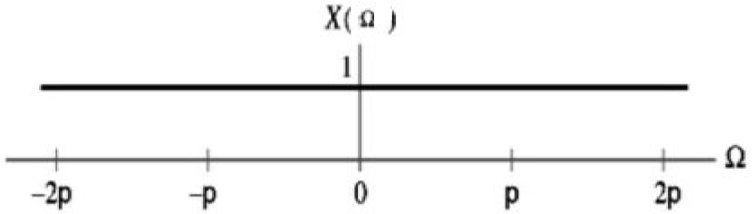
The amplitude and phase spectra are periodic with a period of 2π and thus the ***frequency range*** of any discrete signal is limited to the range (–π, π] or (0,2π].

Example: find DTFT for $x(n)=\delta(n)$

Solution: $$X(\Omega)=\sum_{n=-\infty}^{\infty} x[n]\ e^{-j\Omega n}=\sum_{n=-\infty}^{\infty} \delta[n]\ e^{-j\Omega n}=e^{-j\Omega(0)}=1$$

$$\delta[n] \overset{DTFT}{\Longrightarrow} 1$$





Example

Find the DTFT of the sequence $f(n) = 0.8^n$ for $n = 0,1,2,3\ldots$

Solution

$$F(\Omega)=\mathcal{F}[x[n]]=\sum_{n=-\infty}^{\infty} f(n)\ e^{-j\Omega n}$$

$$F(\Omega)=\sum_{n=0}^{\infty} 0.8^n e^{-j\Omega n}=\sum_{n=0}^{\infty}\left(0.8e^{-j\Omega}\right)^n=\frac{1}{1-0.8e^{-j\Omega}}$$

$$|F(\Omega)|=\frac{1}{\sqrt{1.64-1.6\cos\Omega}}\ ;\ Arg\,F(\Omega)=\tan^{-1}\left(\frac{0.8\sin\Omega}{1-0.8\cos\Omega}\right)$$

If we set $\Omega = -\Omega$ in the last two equations we find that the amplitude is an even function and the argument is an odd function.

Example: find DTFT for $x(n) = \alpha^n \, u[n] \qquad \alpha < 1$

Solution: $$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \alpha^n\, u[n] e^{-j\Omega n} = \sum_{n=0}^{\infty} (\alpha\, e^{-j\Omega})^n = \frac{1}{1-\alpha\, e^{-j\Omega}}$$

This a geometric progression with a common ratio $\alpha\, e^{-j\Omega}$, therefore

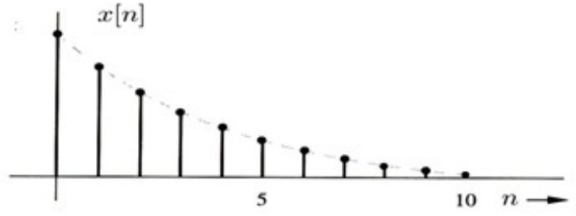
$$X(\Omega) = \frac{1}{1-\alpha\, e^{-j\Omega}} = \frac{1}{1-\alpha\cos\Omega + j\alpha\sin\Omega}$$

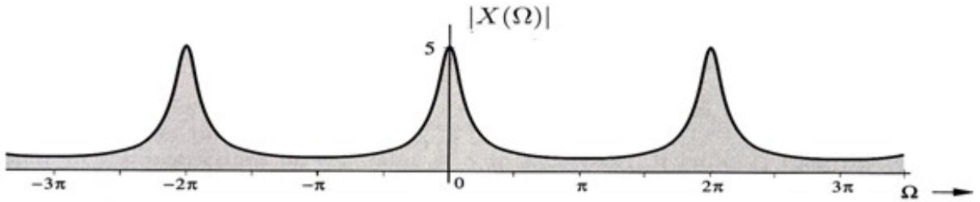
$$|X(\Omega)| = \frac{1}{\sqrt{(1-\alpha\cos\Omega)^2 + (\alpha\sin\Omega)^2}} = \frac{1}{\sqrt{1+\alpha^2-2\alpha\cos\Omega}}$$

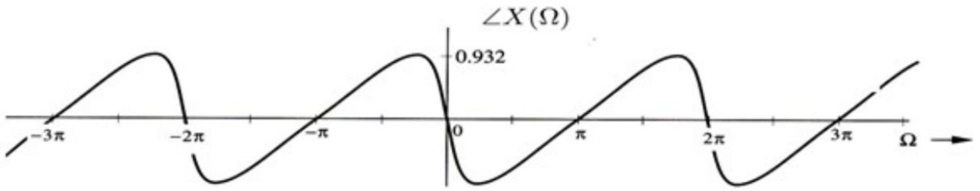
The phase is given by

$$arg(X(\Omega)) = -\tan^{-1}\frac{\alpha\sin\Omega}{1-\alpha\cos\Omega}$$

Figures plotted for $\alpha = 0.8$

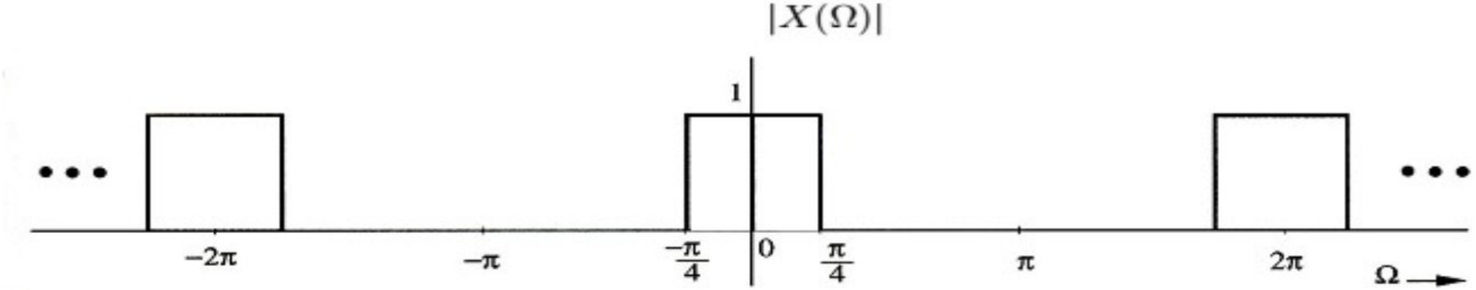






Note that the spectra are continuous and periodic functions of **Ω** with the period 2π. The magnitude spectrum is an even function and the phase spectrum is an odd function of **Ω**

Example: Find the inverse DT Fourier transform of the rectangular pulse spectrum shown in Figure below. Plot the time domain signal.



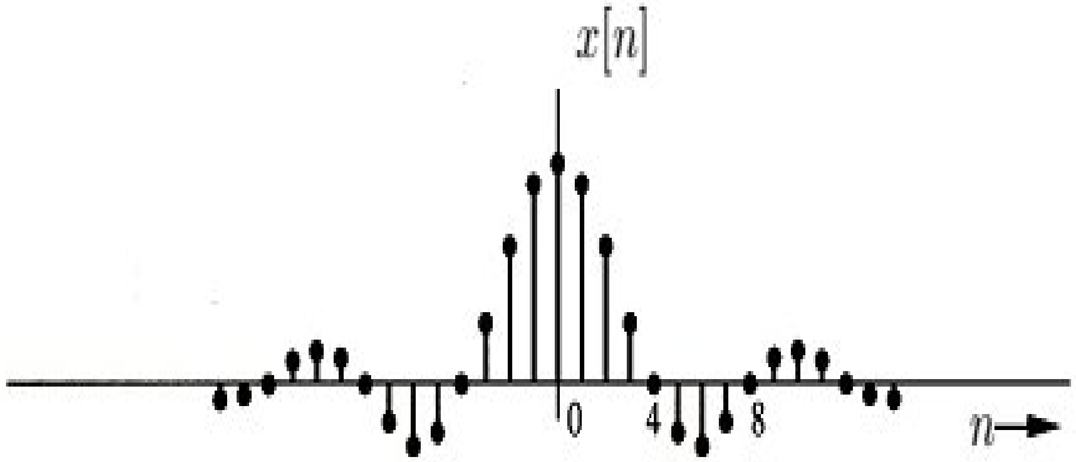
Solution:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j\Omega n}d\Omega$$

$$= \frac{1}{j2\pi n}e^{j\Omega n}\Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n} = \frac{1}{4}\text{sinc}\left(\frac{\pi n}{4}\right)$$



Basic properties of the Fourier transform are presented in the following. There are many similarities and several differences from the continuous-time case.

Periodicity: $X(\Omega + 2\pi) = X(\Omega)$

As a consequence of this equation, in the DT case we have consider values of Ω (rad) only over the range $0 \le \Omega < 2\pi$ or $-\pi \le \Omega < \pi$, while in CT case we have to consider ω*(rad/sec)* over the entire range $-\infty < \omega < \infty$

Linearity: $a_1 x_1[n] + a_2 x_2[n] \underset{DTFT}{\Longleftrightarrow} a_1 X_1(\Omega) + a_2 X_2(\Omega)$

Time Shifting: $x[n-n_0] \underset{DTFT}{\Longleftrightarrow} e^{-j\Omega n_0} X(\Omega)$

Frequency Shifting: $e^{-jn\Omega_0} x[n] \underset{DTFT}{\Longleftrightarrow} X(\Omega-\Omega_0)$

Conjugation : $x^*[n] \underset{DTFT}{\Longleftrightarrow} X^*(-\Omega)$

where * denotes the complex conjugate.

Time Reversal: $x[-n] \underset{DTFT}{\Longleftrightarrow} X(-\Omega)$

Time Scaling:

As mentioned before, the scaling property of a continuous-time Fourier transform is expressed as

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

However, in the discrete-time case, *x[an]* is not a sequence if *a* is not an integer. On the other hand, if *a* is an integer, say *a* = 2, then *x[2n]* consists of only the even samples of *x[n]*. Thus, time scaling in discrete time takes on a form somewhat different from above equation.

Let *m* be a positive integer and define the sequence

$$x_{(m)}[n] = \begin{cases} x[n/m] = x[k] & \text{if } n = km,\ k = \text{integer} \\ 0 & \text{if } n \neq km \end{cases}$$

Then we have $x_{(m)}[n] \underset{DTFT}{\Longleftrightarrow} X(m\Omega)$

This equation is the discrete-time counterpart of continuous time case. It states again the inverse relationship between time and frequency. That is, as the signal spreads in time ($m > 1$), its Fourier transform is compressed. Note that $X(m\Omega)$ is periodic with period $2\pi/m$ since $X(\Omega)$ is periodic with period 2π.

Differentiation in Frequency:

$$n\,x[n] \underset{DTFT}{\Longleftrightarrow} j\frac{dX(\Omega)}{d\Omega}$$

Differencing

$$x[n]-x[n-1] \underset{DTFT}{\Longleftrightarrow} \left(1-e^{-j\Omega}\right)X(\Omega)$$

The sequence we $x[n]-x[n-1]$ is called the first difference sequence. Equation above is easily obtained from the linearity and time shifting properties.

Accumulation

$$\sum_{k=-\infty}^{n} x[k] \underset{DTFT}{\Longleftrightarrow} \pi X(0)\,\delta(\Omega)+\frac{1}{1-e^{-j\Omega}}X(\Omega) \qquad |\Omega|\le\pi$$

Note that accumulation is the DT counterpart of integration. The impulse term on the right hand side in above Eq. reflects the DC or average value that can result from the accumulation.

Convolution

$$x_1[n] * x_2[n] \underset{DTFT}{\Longleftrightarrow} X_1(\Omega)\,X_2(\Omega)$$

The convolution in time domain is the multiplication in frequency domain. The convolution plays an important role in the study of discrete-time LTI systems.

Multiplication

$$x_1[n]\,x_2[n] \underset{DTFT}{\Longleftrightarrow} X_1(\Omega) * X_2(\Omega)$$

Where * denotes the periodic convolution defined by

$$X_1(\Omega) * X_2(\Omega)=\int_{2\pi} X_1(\theta)\,X_2(\Omega-\theta)\,d\theta$$

Common DTFT Pairs

$x[n]$	$X(\Omega)$
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\Omega n_0}$
$x[n]=1$	$2\pi\delta(\Omega), \lvert\Omega\rvert \le \pi$
$e^{j\Omega_0 n}$	$2\pi\delta(\Omega-\Omega_0), \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$\cos \Omega_0 n$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$\sin \Omega_0 n$	$-j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$u[n]$	$\pi\delta(\Omega)+\frac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert \le \pi$
$-u[-n-1]$	$-\pi\delta(\Omega)+\frac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert \le \pi$
$a^n u[n], \lvert a\rvert < 1$	$\frac{1}{1-ae^{-j\Omega}}$
$-a^n u[-n-1], \lvert a\rvert > 1$	$\frac{1}{1-ae^{-j\Omega}}$
$(n+1)a^n u[n], \lvert a\rvert < 1$	$\frac{1}{(1-ae^{-j\Omega})^2}$
$a^{\lvert n\rvert}, \lvert a\rvert < 1$	$\frac{1-a^2}{1-2a\cos\Omega+a^2}$
$x[n]=\begin{cases}1 & \lvert n\rvert \le N_1 \\ 0 & \lvert n\rvert > N_1\end{cases}$	$\frac{\sin\left[\Omega\left(N_1+\frac{1}{2}\right)\right]}{\sin(\Omega/2)}$
$\frac{\sin Wn}{\pi n}, 0 < W < \pi$	$X(\Omega)=\begin{cases}1 & 0 \le \lvert\Omega\rvert \le W \\ 0 & W < \lvert\Omega\rvert \le \pi\end{cases}$
$\sum_{k=-\infty}^{\infty} \delta[n-kN_0]$	$\Omega_0 \sum_{k=-\infty}^{\infty} \delta(\Omega-k\Omega_0), \Omega_0=\frac{2\pi}{N_0}$

Properties of the Discrete-Time Fourier Transform

Property	Sequence	Fourier transform
	$x[n]$	$X(\Omega)$
	$x_1[n]$	$X_1(\Omega)$
	$x_2[n]$	$X_2(\Omega)$
Periodicity	$x[n]$	$X(\Omega+2\pi)=X(\Omega)$
Linearity	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1(\Omega)+a_2X_2(\Omega)$
Time shifting	$x[n-n_0]$	$e^{-j\Omega n_0}X(\Omega)$
Frequency shifting	$e^{j\Omega_0 n}x[n]$	$X(\Omega-\Omega_0)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Time reversal	$x[-n]$	$X(-\Omega)$
Time scaling	$x_{(m)}[n]=\begin{cases}x[n/m] & \text{if } n=km \\ 0 & \text{if } n \ne km\end{cases}$	$X(m\Omega)$
Frequency differentiation	$nx[n]$	$j\frac{dX(\Omega)}{d\Omega}$
First difference	$x[n]-x[n-1]$	$(1-e^{-j\Omega})X(\Omega)$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\pi X(0)\delta(\Omega)+\frac{1}{1-e^{-j\Omega}}X(\Omega)$ $\lvert\Omega\rvert \le \pi$
Convolution	$x_1[n]*x_2[n]$	$X_1(\Omega)X_2(\Omega)$
Multiplication	$x_1[n]x_2[n]$	$\frac{1}{2\pi}X_1(\Omega)\otimes X_2(\Omega)$

Parseval's relations

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2[n]=\frac{1}{2\pi}\int_{2\pi} X_1(\Omega)X_2(-\Omega)\,d\Omega$$

$$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2=\frac{1}{2\pi}\int_{2\pi} \lvert X(\Omega)\rvert^2\,d\Omega$$

Fourier Transform Properties

Property Name	Property	
Linearity	$ax(t)+bv(t)$	$aX(\omega)+bV(\omega)$
Time Shift	$x(t-c)$	$e^{-j\omega c}X(\omega)$
Time Scaling	$x(at), \quad a \neq 0$	$\frac{1}{a}X(\omega/a), \quad a \neq 0$
Time Reversal	$x(-t)$	$X(-\omega)$
Multiply by t^n	$t^n x(t), \quad n=1,2,3,\ldots$	$j^n \frac{d^n}{d\omega^n}X(\omega), \quad n=1,2,3,\ldots$
Multiply by Complex Exponential	$e^{j\omega_o t}x(t), \quad \omega_o$ real	$X(\omega-\omega_o), \quad \omega_o$ real
Multiply by Sine	$\sin(\omega_o t)x(t)$	$\frac{j}{2}[X(\omega+\omega_o)-X(\omega-\omega_o)]$
Multiply by Cosine	$\cos(\omega_o t)x(t)$	$\frac{1}{2}[X(\omega+\omega_o)+X(\omega-\omega_o)]$
Time Differentiation	$\frac{d^n}{dt^n}x(t), \quad n=1,2,3,\ldots$	$(j\omega)^n X(\omega), \quad n=1,2,3,\ldots$
Time Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$
Convolution in Time	$x(t)*h(t)$	$X(\omega)H(\omega)$
Multiplication in Time	$x(t)w(t)$	$\frac{1}{2\pi}X(\omega)*W(\omega)$
Parseval's Theorem (Energy)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2 d\omega$ if $x(t)$ is real $\int_{-\infty}^{\infty}\lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2 d\omega$	
Duality: If $x(t) \leftrightarrow X(\omega)$	$X(t)$	$2\pi x(-\omega)$

Modulation Property

The frequency-shifting / modulation property is one of the most important Fourier transform properties as modulation is the basic operation that underlies all communication systems such as Amplitude Modulation (AM) and Frequency Modulation (FM).

Exercise: Proof that

$$x(t)\, e^{j2\pi f_0 t} \iff X(f - f_0)$$

Mode of Communication:

- **Broadcasting**

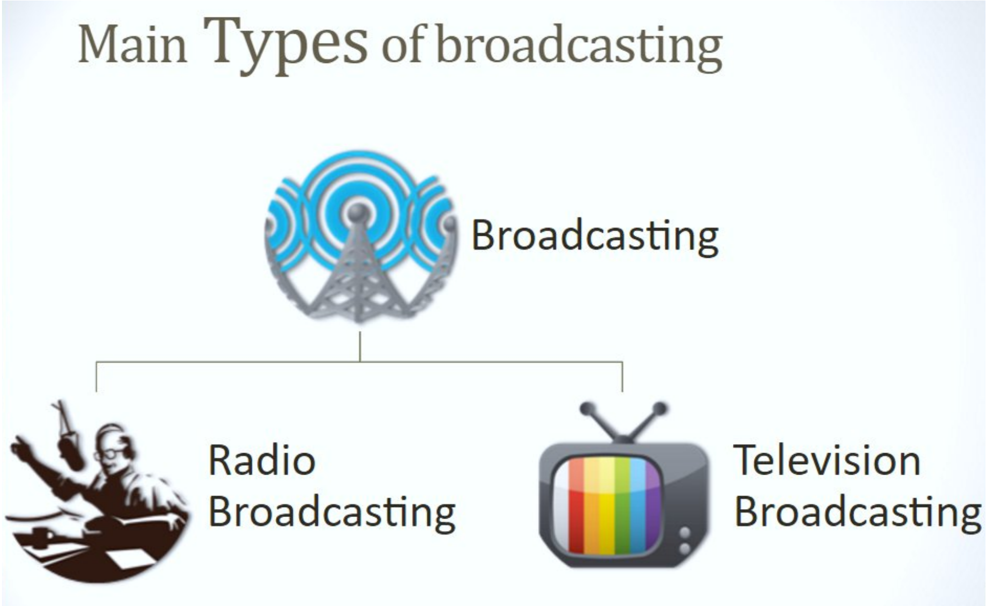
Involves the use of a single powerful transmitter transmit to many receivers.

Demodulation takes place in the receiver.

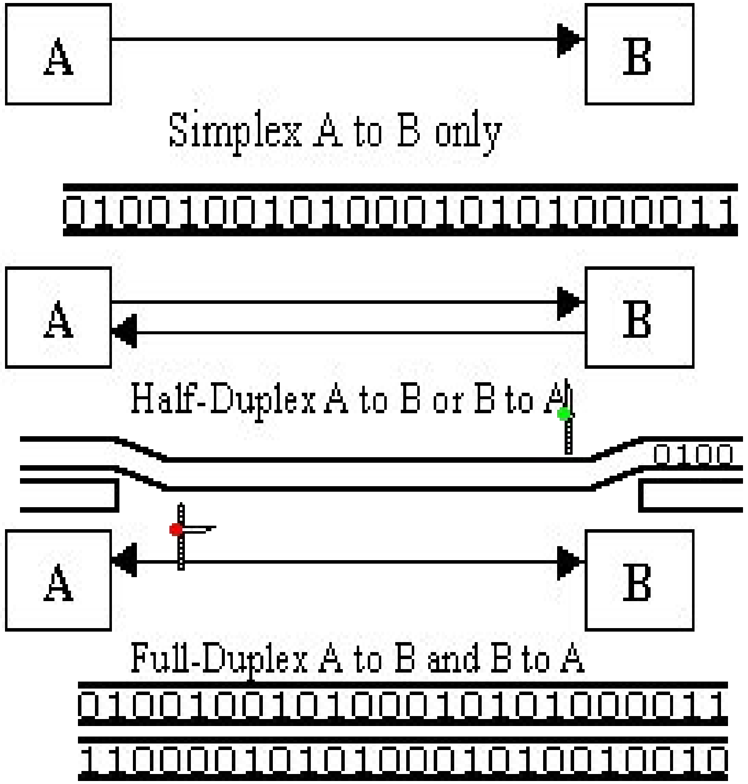
Information-bearing signals flow in one direction which is called **Simplex communication.**

Simplex communication is a communication channel that sends information in one direction only.

Examples are **TV and radio**



Mode of Communication:



➢ Point to point Communication

Where a communication process takes place over a link between a single transmitter and a receiver.

- Information-bearing signals flow in bidirectional, which requires the use of a transmitter and receiver at each end of the link

Tow types of point to point communication

In a **full-duplex system**, both parties can communicate with each other simultaneously. An example of a full-duplex device is a telephone; the parties at both ends of a call can speak and be heard by the other party simultaneously because there are two communication paths/channels between them.

In a **half-duplex system**, there are still two clearly defined paths/channels, and each party can communicate with the other but not simultaneously; the communication is one direction at a time. An example of a half-duplex device is a walkie-talki radio.