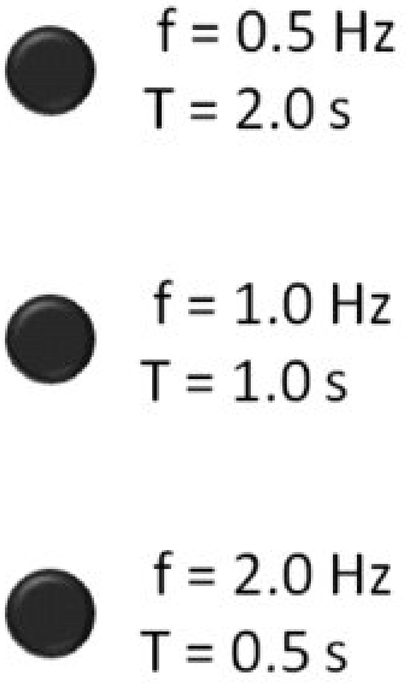
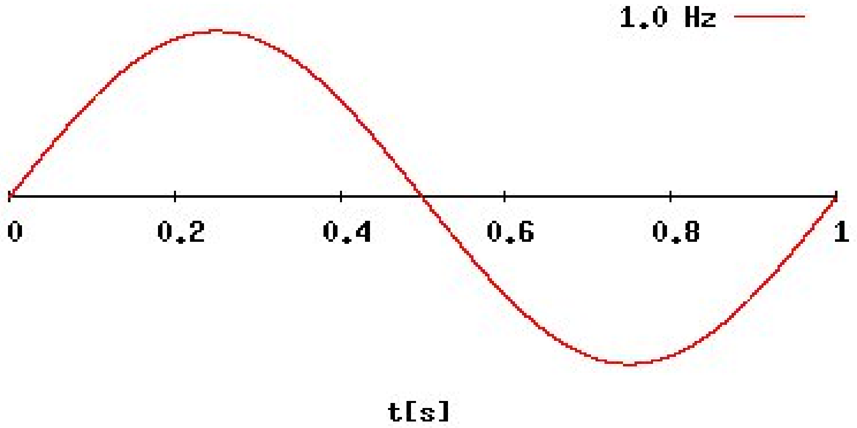
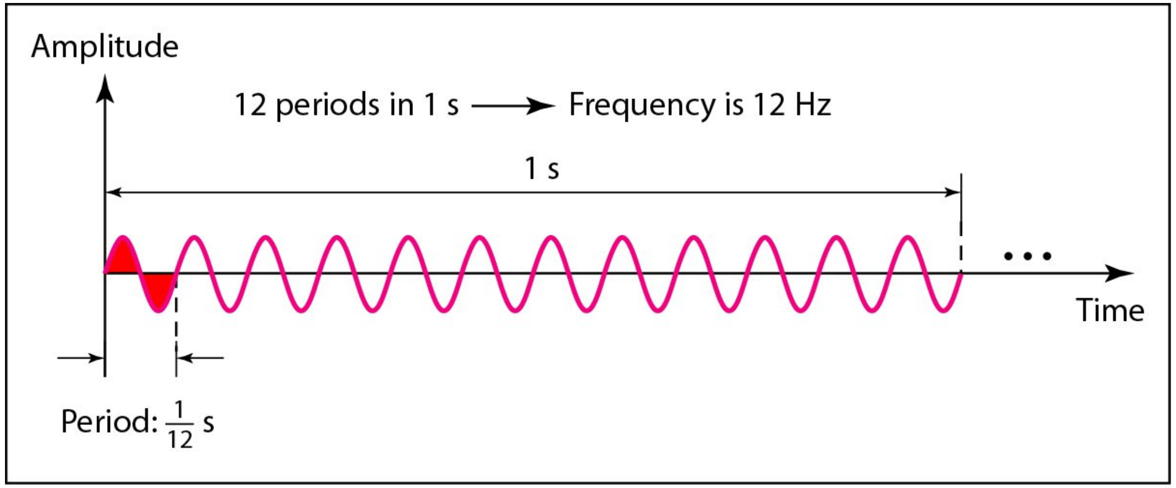
- **Frequency** is the rate of change with respect to time.
- Change in a short span of time means high frequency.
- Change over a long span of time means low frequency.

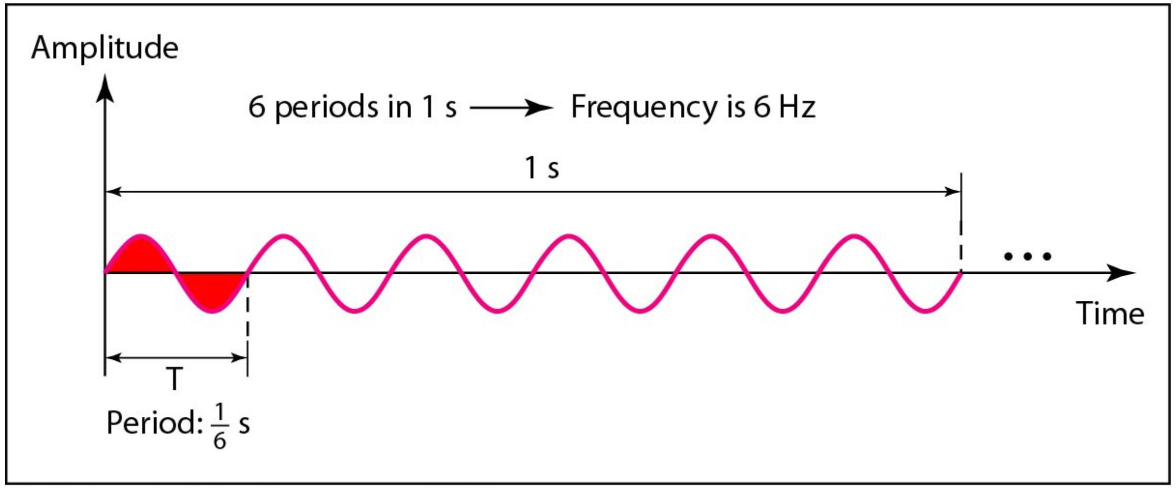




Example ***Two signals with the same amplitude and phase, but different frequencies***



a. A signal with a frequency of 12 Hz

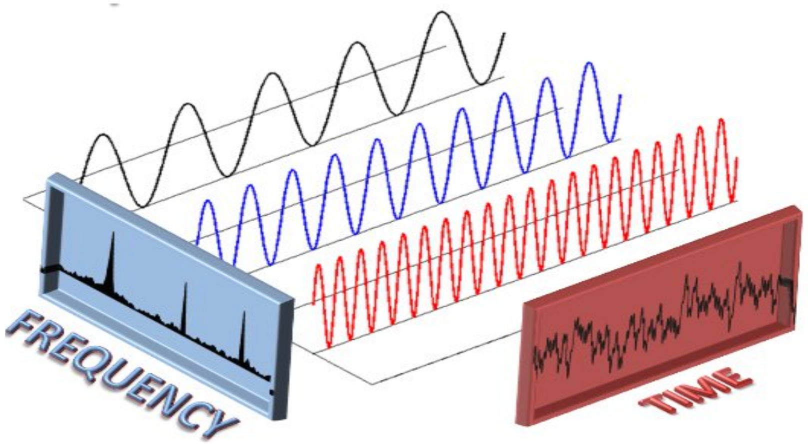


b. A signal with a frequency of 6 Hz

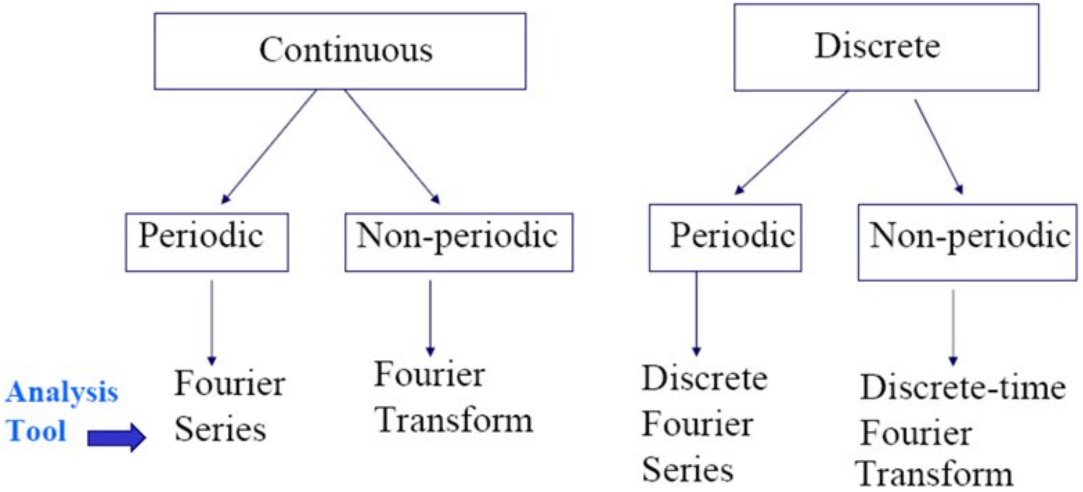
Fourier Analysis

The subject of Fourier analysis is essential for describing certain types of systems and their properties in frequency domain. It is concerned with representing a signal as weighted superposition of complex sinusoids. The complex sinusoids are basic signals that can be used to construct a broad and useful class of signals. There are four distinct Fourier representations:

- Continuous-time Fourier series,
- Discrete-time Fourier series,
- Continuous-time Fourier transform,
- Discrete-time Fourier transform.



Fourier Series (FS) applies to continuous-time periodic signals. Discrete Time Fourier Series (DTFS) applies to discrete-time periodic signals. Nonperiodic signals have Fourier Transform (FT) representation. Discrete Time Fourier Transform (DTFT) applies to a signal that is discrete in time and non-periodic.



Fourier series converts the time-domain signals into frequency-domain (**or spectral**) representations. Fourier series is a means to represent a ***periodic signal*** as an infinite sum of sine wave components. A periodic signal is just a signal that repeats its pattern at some period. The primary reason that we use Fourier series is that we can better analyze a signal in another domain rather in the original domain. A continuous-time signal ***x(t)*** to be periodic if there is a positive nonzero value of ***T*** for which

$$x(t+T) = x(t) \qquad \text{all } t$$

The fundamental period T_o of ***x(t)*** is the smallest positive value of ***T*** for which above Eq. is satisfied, and $1/T_o = f_o$ is referred to as the fundamental frequency. Two basic examples of periodic signals are the real sinusoidal signal and the complex exponential signal

$$\boxed{x(t) = A\cos(\omega_0 t + \varphi) \qquad ; \qquad x(t) = e^{j\omega_0 t}}$$

where $\omega_o = 2\pi/T_o = 2\pi f_o$ is called the fundamental angular frequency.

Review Example/

What are the magnitude, frequency and phase of the following sinusoidal signal?

x(t)=10 cos (10πt+π/4)?

Solution: A**=10** ; $f = 5$ ***Hz*** **,** $\varphi = \pi/4$,

Complex Exponential Fourier Series Representation:

The complex exponential Fourier series representation of a periodic signal ***x(t)*** with fundamental period T_o is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{jk\omega t}$$

$$\omega = \frac{2\pi}{T}$$

where c_k are known as the complex Fourier coefficients and are given by

$$\boxed{c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega t}\, dt}$$

Setting *k=0;*

$$c_0 = \frac{1}{T}\int_{0}^{T} x(t)\, dt$$

which indicates that c_0 equals the average value of ***x (t)*** over a period. When ***x(t)*** is real, then

$$c_{-k} = c^{*}{}_{k}$$

where the asterisk indicates the complex conjugate.

To find the Fourier coefficients of periodic *x(t):*

1. - For a given *k*, multiply ***x(t)*** by $e^{-jk\omega t}$, and take the area under the curve (dividing by **T**).
2. - Repeat step (1) for all ***k***.

Example : Determine the complex exponential Fourier series representation for the following signal: ***x(t)= cos(ωt)***

Solution : we can write the ***cos*** signal as below and compare it with general equation

$$\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) = \frac{1}{2}e^{-j\omega_0 t} + \frac{1}{2}e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

The complex Fourier coefficients for ***cos(ωt)*** are

$$c_1 = \frac{1}{2} \qquad , c_{-1} = \frac{1}{2} \qquad , \; c_k = 0 \text{ for } k \neq \pm 1$$

Example : Find FS for ***x(t)= sin(3πt+ π/2)***

$$x(t) = \sin(3\pi t + \frac{\pi}{2}) = \frac{e^{j(3\pi t + \frac{\pi}{2})} - e^{-j(3\pi t + \frac{\pi}{2})}}{2j}$$

$$= \frac{1}{2j}e^{j3\pi t}e^{j\frac{\pi}{2}} - \frac{1}{2j}e^{-j3\pi t}e^{-j\frac{\pi}{2}}$$

Since:

- $e^{j\frac{\pi}{2}} = j$
- $e^{-j\frac{\pi}{2}} = -j$

We have:

$$\frac{1}{2j}(j)e^{j3t\pi} - \frac{1}{2j}(-j)e^{-j3t\pi} \rightarrow \frac{1}{2}e^{(1)j3t\pi} + \frac{1}{2}e^{(-1)j3t\pi}$$

Compare with $\sum_{k=-\infty}^{\infty} c_k\, e^{jk\omega t}$

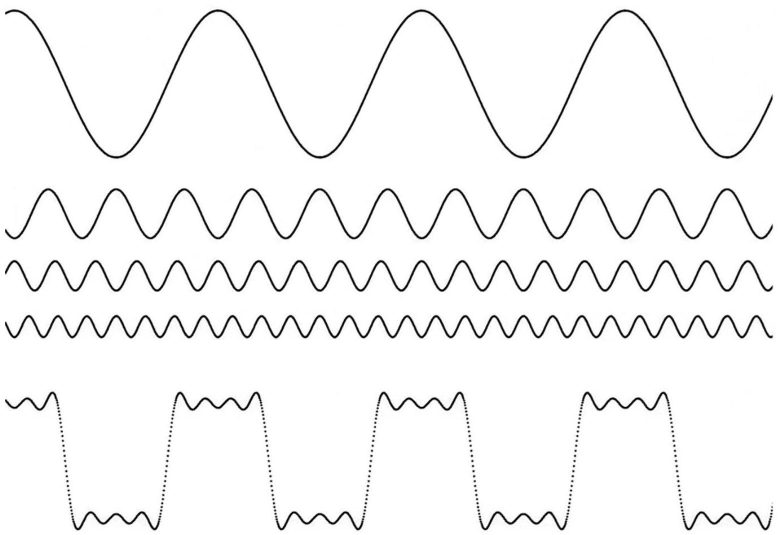
$$c_{-1} = \frac{1}{2}, \; c_1 = \frac{1}{2}, \quad c_k = 0 \,, \; k \neq -1, 1$$

Note that : $x(t)=\sin(3\pi t+\pi/2)=\cos(3\pi t)$.

Trigonometric Fourier Series:

The trigonometric Fourier series representation of a periodic signal ***x(t)*** with fundamental period ***T*****,** is given by

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos k\omega t + b_k \sin k\omega t \right) \qquad \omega = \frac{2\pi}{T}$$



where a_k, and b_k, are the Fourier coefficients given by

$$a_0 = \frac{2}{T}\int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T}\int_0^T x(t) \cos k\omega t\, dt \qquad b_k = \frac{2}{T}\int_0^T x(t) \sin k\omega t\, dt$$

The coefficients a_k, and b_k, and the complex Fourier coefficients c_k, are related by

$$c_0 = \frac{a_0}{2} \quad , \qquad a_k = c_k + c_{-k} \qquad , \quad b_k = j(c_k - c_{-k})$$

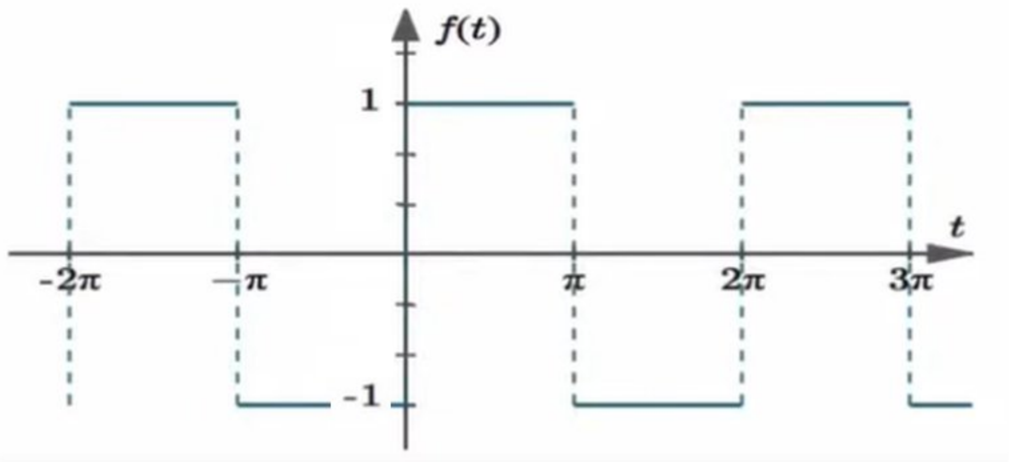
From these equations we can obtain

$$c_k = \frac{1}{2}(a_k - jb_k) \quad and \quad c_{-k} = \frac{1}{2}(a_k + jb_k)$$

When ***x(t)*** is real, then a_k, and b_k, are real and we have

$$a_k = 2\, Re[c_k]\ , \quad b_k = -2\, Im[c_k]$$

Example: Find the Complex Exponential **FS** coefficients for ***f(t)*** shown



Solution:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (\, a_k \cos k\omega t + b_k \sin k\omega t)$$

This wave repeats every T= 2π seconds.

$$a_0 = \frac{2}{T}\int_0^T x(t)\, dt = \frac{2}{2\pi}\int_0^{2\pi} x(t)\, dt$$

Given that f(t) is symmetric around zero, the positive and negative areas cancel out.
So: $\mathbf{a_0=0}$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos k\omega t\, dt \quad = \frac{1}{\pi}\left(\int_0^{\pi} 1\cdot\cos(kw\cdot t)\, dt + \int_{\pi}^{2\pi} -1\cdot\cos(kw\cdot t)\, dt\right)$$

This results a_k=0 for all kw (by symmetry and integration).

$$b_k = \frac{2}{T}\int_0^T x(t)\sin k\omega t\, dt = 2 \quad \frac{1}{\pi}\left(\int_0^{\pi} 1\cdot\sin(kw\cdot t)\, dt + \int_{\pi}^{2\pi} -1\cdot\sin(kw\cdot t)\, dt\right)$$

Even and Odd Signals:

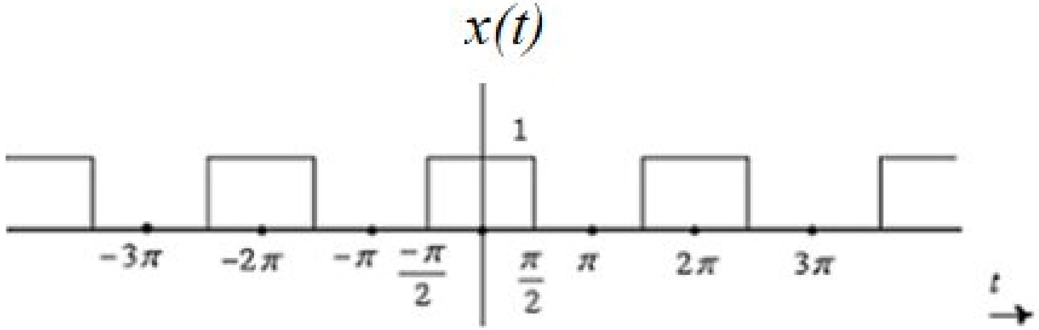
If a periodic signal ***x(t)*** is even, then $b_k = 0$ and its Fourier series contains only cosine terms:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega t \qquad \omega = \frac{2\pi}{T}$$

If ***x(t)*** is odd, then $a_k = \mathbf{0}$ and its Fourier series contains only sine terms:

$$x(t) = \sum_{k=1}^{\infty} b_k \sin k\omega t \qquad \omega = \frac{2\pi}{T}$$

Example: Find the Fourier Series for ***x(t)***



Solution : ***x(t)*** is even, then $b_k = 0$

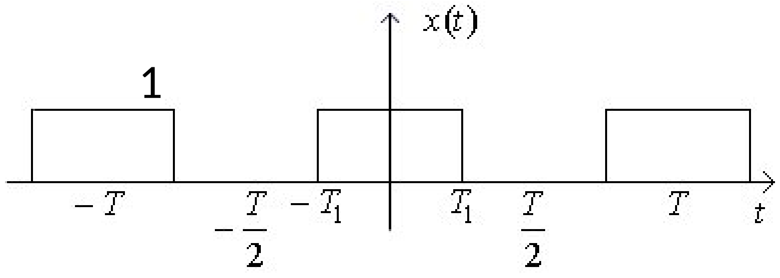
T=2π , ω=1

$$a_0 = \frac{2}{T}\int_0^T x(t)dt \quad \Rightarrow a_0 = \frac{1}{T}\int_{-T_o/2}^{T_o/2} dt = 1$$

$$a_k = \frac{2}{2\pi}\int_{-\pi/2}^{\pi/2} x(t)\cos kt\, dt \quad = \frac{2}{k\pi}\sin\left(\frac{k\pi}{2}\right)$$

$$\Rightarrow x(t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2\sin(\frac{\pi k}{2})}{\pi k}\cos kt$$

Example: Find the Complex Exponential **FS** coefficients for ***x(t)*** shown



Solution:

$$\boxed{c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega t}\, dt}$$

$$C_k = \frac{1}{T}\int_{-T_1}^{T_1} (1)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\left(\frac{-1}{jk\omega_0}\right) e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}$$

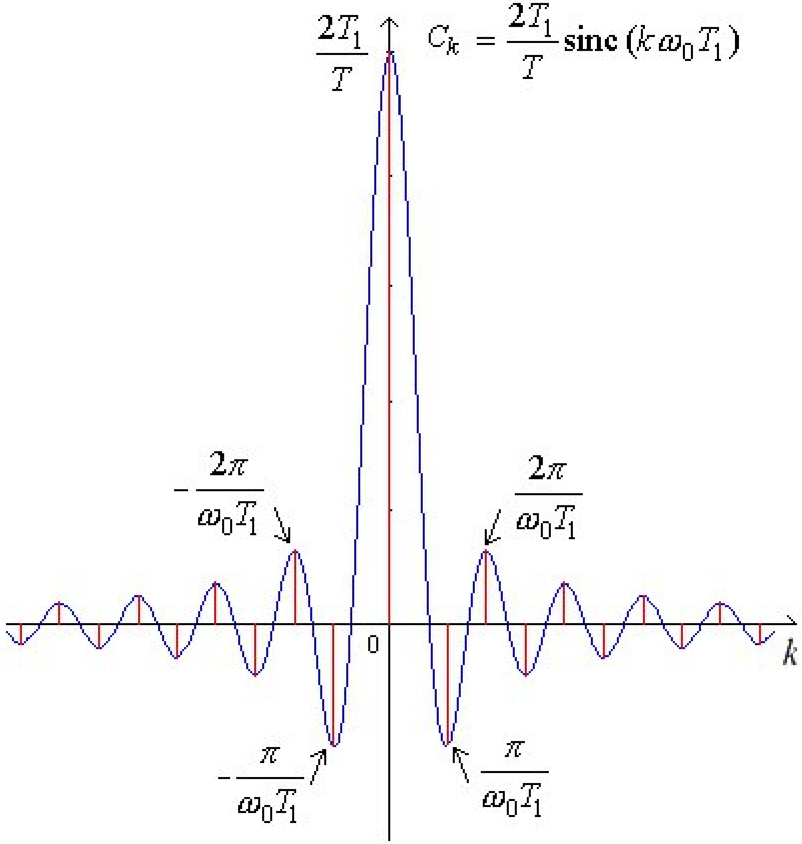
$$= \frac{-1}{jk\omega_0 T}\left(e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}\right)$$

$$= \frac{2}{k\omega_0 T} * \frac{1}{2j}\left(e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right)$$

$$= \frac{2}{k\omega_0 T}\sin(k\omega_0 T_1)$$

$$= \frac{2T_1}{T}\frac{\sin(k\omega_0 T_1)}{k\omega_0 T_1}$$

$$= \frac{2T_1}{T}\mathrm{sinc}(k\omega_0 T_1)$$



Harmonic Form Fourier Series:

Another form of the Fourier series representation of a real periodic signal ***x(t)*** with fundamental period ***T*** is the harmonic form Fourier series of ***x(t)*** which is given by

$$x(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos(k\omega t - \theta_k) \qquad \omega = \frac{2\pi}{T}$$

The term C_o is known as the dc component, and the term $C_o \cos(k\omega t - \theta_k)$ is referred to as the *kth* harmonic component of $x(t)$. The first harmonic component $C_1 \cos(\omega t - \theta_1)$ is commonly called the ***fundamental component*** because it has the same fundamental period as $x(t)$. The coefficients C_k and the angles θ_k are called the harmonic amplitudes and phase angles, respectively, and they are related to the Fourier coefficients a_k, and b_k, by

$$C_0 = \frac{a_0}{2} \qquad C_k = \sqrt{{a_k}^2 + {b_k}^2} \qquad \theta_k = \tan^{-1}\frac{b_k}{a_k}$$

Note : The FS coefficients are known as a frequency domain representation of ***x(t)*** because it is the coefficient associated with complex sinusoid at frequency ***kω.***

Amplitude and Phase Spectra of a Periodic Signal

The complex form for Fourier series shown below is more general and usually more convenient

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} C_k \, e^{jk\omega t}} \qquad \omega = \frac{2\pi}{T}$$

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega t}\, dt$$

Let the complex Fourier coefficients C_k in above Eq. be expressed as

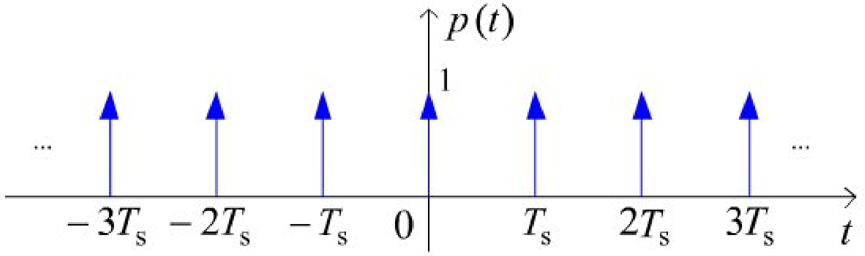
$$\boxed{C_k = |C_k| \; e^{j\varphi_k}}$$

Where a plot of $|\, C_k \,|$ versus the angular frequency $\boldsymbol{\omega}$ is called the ***amplitude spectrum*** of the periodic signal ***x (t)*** ,and a plot of $\boldsymbol{\varphi}$ versus $\boldsymbol{\omega}$ is called the ***phase spectrum*** of ***x(t)***. Since the index ***k*** assumes only integers, the amplitude and phase spectra are not continuous curves but appear only at the discrete frequencies $\boldsymbol{k\omega}$. They are therefore referred to as **discrete frequency spectra or line spectra**. For a real periodic signal ***x(t)*** we have $C_{-k} = C_k{}^*$. Thus

$$\boxed{|C_{-k}| = |C_k|} \qquad \boxed{\varphi_{-k} = -\,\varphi_k}$$

Hence, the **amplitude spectrum** is an **even** function of ***ω,*** and the **phase spectrum** is an **odd** function of ***ω*** for a real periodic signal.

Example Calculate the FS coefficients for the impulse train and plot the magnitude of its frequency spectrum (which are simply the Fourier Series coefficients). $p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$

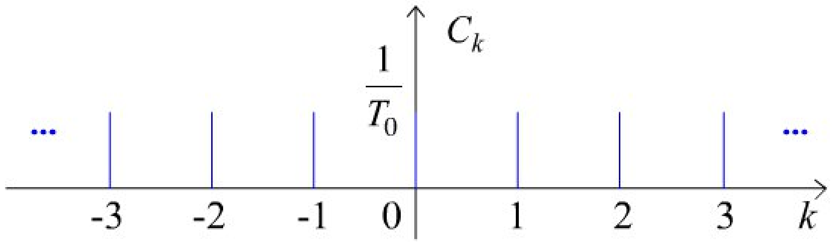


Solution

$$C_k = \frac{1}{T_0} \int_{T_0} p(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t)\, e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T_0} = C_k, \forall k$$



Power Content of a Periodic Signal: Parseval's theorem

As mentioned before, the average power of a periodic signal x(t) over any period can be expressed as

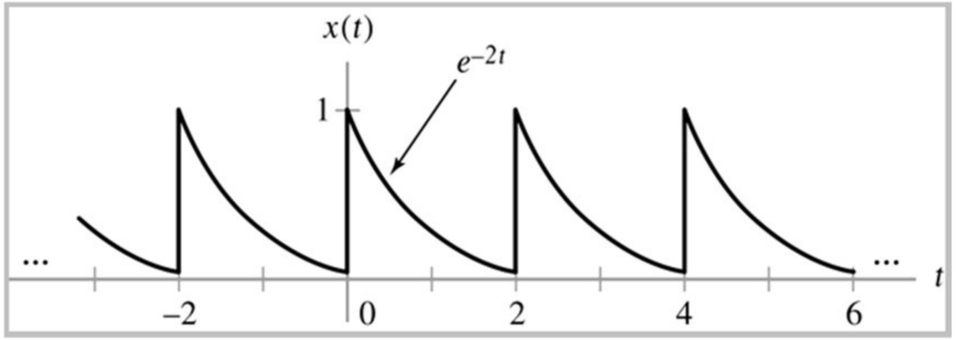
$$\boxed{\boldsymbol{P = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt}}$$

If ***x(t)*** is represented by the complex exponential Fourier series, then it can be shown that

$$\boxed{\boldsymbol{\frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |C_k|^2}}$$

Above equation is called **Parseval's identity (or Parseval's theorem**) for the Fourier series.

Example/ Determine the FS coefficients for the signal *x(t)* depicted in Figure below. Derive analytical expressions and plot, the double-sided magnitude, phase, and power spectra.



Solution:

Step 1: Determine $\boldsymbol{T}$ and $\boldsymbol{\omega}$.

Step 2: Solve for C_k.

Step 3: Plot magnitude, phase and power spectrum

$\boldsymbol{T} = 2$ sec, $\boldsymbol{\omega = 2\pi/T = \pi}$ *rad*

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega t}\, dt = \frac{1}{2}\int_0^2 e^{-2t}\, e^{-jk\pi t}\, dt = \frac{1}{2}\int_0^2 e^{-(2+jk\pi)t}\, dt$$

$$C_k = \frac{1}{2}\left[\left\{\frac{-\,e^{-(2+jk\pi)t}}{2+jk\pi}\right\}_0^2\right] = \frac{1}{2}\left[\left\{\frac{-\,e^{-4-j2k\pi}+1}{2+jk\pi}\right\}\right] = \frac{1-\,e^{-4}}{4+j2k\pi} \qquad \text{since we have} \quad e^{\pm j2k\pi} = 1$$

Analytical expressions for magnitude and phase spectra:

$$C_k = \frac{1-\,e^{-4}}{4+j2k\pi} \times \frac{4-j2k\pi}{4-j2k\pi} = \frac{\left(4-4e^{-4}\right) + j(e^{-4}2\pi k-2\pi k)}{4^2+(2k\pi)^2}$$

$$= \frac{\left(4-4e^{-4}\right)}{4^2+(2k\pi)^2} + j\,\frac{(e^{-4}2\pi k-2\pi k)}{4^2+(2k\pi)^2}$$

Note that $|a \pm jb| = \sqrt{a^2+b^2}$

$$Magnitude = |C_k| = \sqrt{Re^2 + Im^2} = \left|\frac{(1-e^{-4})}{4 + j2k\pi}\right| = \frac{|1-e^{-4}|}{\sqrt{4^2+(2k\pi)^2}}$$

$$Phase = \varphi_k = tan^{-1}(\frac{Im}{Re}) = tan^{-1}\left(\frac{e^{-4}2\pi k - 2\pi k}{(4-4e^{-4})}\right) = tan^{-1}(\frac{-2\pi k * 1 = e^{-4})}{4(1-e^{-4})}) = tan^{-1}(\frac{\pi k}{2})$$

Magnitude spectrum $|c_k|$
Phase spectrum arg{ c_k }= tan⁻¹ (c_k) = tan⁻¹ (2πk/4)
Power spectrum $|c_k|^2$

k	$\|c_k\|$	ϕ_{rad}	$\|c_k\|^2$
-5	0.0310	1.4441	0.0010
-4	0.0386	1.4130	0.0015
-3	0.0509	1.3617	0.0026
-2	0.0744	1.2626	0.0055
-1	0.1318	1.0039	0.0174
0	0.2454	0	0.0602
1	0.1318	-1.0039	0.0174
2	0.0744	-1.2626	0.0055
3	0.0509	-1.3617	0.0026
4	0.0386	-1.4130	0.0015
5	0.0310	-1.4441	0.0010